Theoretical Study of Lattice Solitons in Nonlinear Lumped Networks

by

Samir Husain Abdul-Jauwad

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Abdul-Jauwad, Samir Husain, M.S.

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

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Samir Husain Abdul-Jauwad

under the direction of his Thesis Committee, and approved by all its members, has been presented to and accepted by the Dean of the Graduate School, in partial fulfilment of the requirements for the degree of

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fr Fahd H. Dakhil

Dean of the Graduate School

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Department Chairman

Thesis Committee

Ronald Escott

Chairman

John M. Jarem

Member

Ali Kyrala

Member

بسم الله الرحمن الرحيم

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TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF ILLUSTRATIONS

Figure Page

ABSTRACT

An analytical solution of the equation of motion in an anharmonic one-dimensional lattice was found by Toda. An equivalent solution for a nonlinear lumped LC network was derived as an analogue for the anharmonic one-dimensional lattice. The fundamental properties of solitons were verified. These properties include (1) the dependence of the amplitude of a single soliton on the velocity of propagation (2) the emergence of two solitons after collision without changing their shapes (3) and the conservation of the voltage and power integrals with respect to a normalized time.

INTRODUCTION

It has long been a matter of observation that wave forms of permanent type other than the uniform flows with an undeformed free surface occur in nature; for example Scott-Russel (1) reported in 1844 his observation on what has since been called the solitary wave, which is a wave having a symmetrical form with a single hump and which propagates at uniform velocity without change of form.

In 1895, Korteweg and deVries were able to derive an equation in which they could explain solitary waves (2) for shallow water waves. Later, a number of equations were derived that exhibited solutions of solitary waves nature. Experimental observations gave a rigid ground for the existence of solitary waves and solitons.

In this paper we investigated a nonlinear lumped LC network and observed that the solution is of a solitary wave nature. We also verified using computer calculations the different properties of solitons. These properties included the dependence of the magnitude of a single soliton on the velocity of propagation and the nature of the collision of two solitons.

Chapter one starts with the historical background on the development of solitary waves theory and includes some equations that exhibit solitary wave solutions. It also includes experimental verifications of the theoretical results. Chapter two introduces the existence of a solitary wave solution in a nonlinear lumped LC network. In this chapter we investigate the different properties of solitons using numerical calculations.

Finally, in chapter four we list our conclusions and recommendations for further investigations.

CHAPTER 1

SOLITARY WAVES, STATE OF THE ART

1.1 Introduction

In this chapter we shall attempt to discuss the history and development of solitary waves. This chapter is a key one in the sense of defining solitary waves and particularly solitons as they emerge from some nonlinear dispersive equations.

1.2 Description and Definition of Solitary Waves

The concept of solitary waves was first introduced by Scott-Russel in his "Report on Waves" dated 1844 (1). Scott-Russel was observing the motion of a mass of water caused by the sudden stop of a boat in a narrow water channel. He was able to chase this mass of water which had a height of one to one and a half feet, and found that it propagated as a solitary wave without changing its shape or speed for one or two miles.

In 1895, Korteweg and deVries were able to derive an equation (the KdV equation) in which they could explain this observed phenomena (2) for shallow water waves. This equation includes both nonlinear and dispersive effects but ignores dissipation. A traveling wave solution of the KdV equation can be obtained by assuming

$$\phi(x,t) = \phi_T(\xi) \tag{1.1}$$

where $$\xi = x - ut \tag{1.2}$$

represents the position in a coordinate system moving at a velocity u for which the wave appears stationary.

From Eq. (1.2) we can write

$$\frac{\partial}{\partial x} = \frac{d}{d\xi} \tag{1.3}$$

and

$$\frac{\partial}{\partial t} = - u \frac{d}{d\xi} \tag{1.4}$$

Here we have reduced the original partial differential equation to an ordinary differential equation which can be solved analytically.

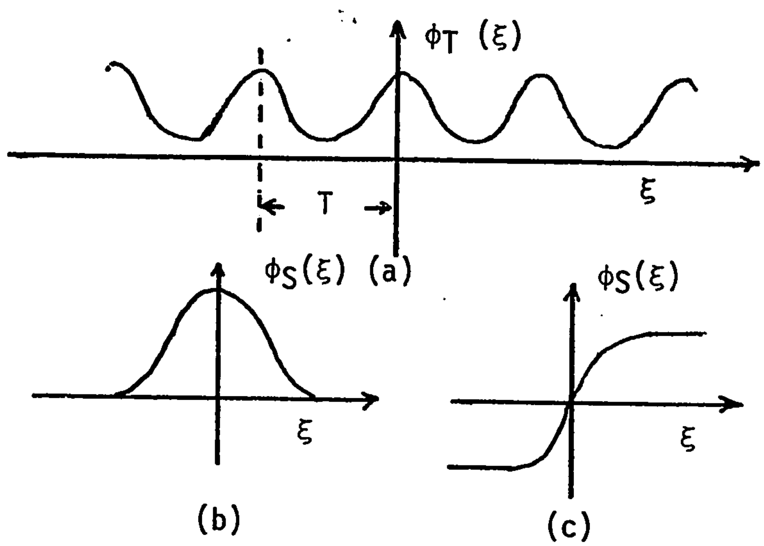

Fig. 1.1 (a) Traveling wave solutions to the KdV equation (b) A solitary wave (c) A solitary wave with different asymptotic values at $\xi = +\infty$ and $\xi = -\infty$.

The qualitative nature of the solution is indicated in Fig.1.1(a). It consists of an infinite and periodic train of localized "humps" for which the velocity u of the train and the period T between the humps are continuously variable. The solitary wave is obtained by letting $T \to \infty$. This single hump is called soliton; which is the special solution of dispersive wave equations. Scott et al (4) gave the following definitions:

Definition: Given an underlying wave equation, a traveling wave $\phi_T(\xi)$ is a solution which depends upon x and t only through $\xi \equiv x-ut$, where u is a fixed constant.

Definition: A solitary wave $\phi_{ST}(\xi)$ is a localized traveling wave; or, more precisely, a traveling wave whose transition from one constant asymptotic state as $\xi \to -\infty$ to (possibly) another as $\xi \to +\infty$ is essentially localized in ξ.

Two types of solitary waves are sketched in Fig.1.1 (b) and (c). The concept of a solitary wave has been restricted to those waves having the general shape of Fig. 1.1(b) and excluding those of Fig.1.1(c). We choose not to make this restriction because the derivative with respect to ξ of the wave in Fig. 1.1(c) has the shape indicated in Fig. 1.1(b).

The solitary wave was long considered unimportant in the mathematical structure of nonlinear wave theory. Since it is a special solution to the partial differential equation, many have assumed that somewhat special initial conditions would be required to launch it and, therefore, that its role in relation to the initial value problem would be a minor one at best. Furthermore, it was generally supposed that if two solitary waves were initially launched on a collision course, the nonlinear interaction upon collision would completely destroy their integrity. With the development of the modern digital computers it became possible to test these assumptions by direct calculations.

The results of the first such test were obtained by Perring and

Skyrme (5). They were interested in the solitary wave solutions of the sine-Gordon equation as a simple model for the elementary particles of matter. Computer experiments were initiated to see how much model elementary particles would scatter upon collision. The computer solution indicated that the solitary waves did not scatter. They emerged from the collision having the same shapes and velocities which they entered. From this computer studies, Perring and Skyrme were able to find analytic expressions expressing collision events.

Shortly thereafter Zabusky and Kruskal published results of completely independent computer study of the application of the KdV equation to the investigation of plasma waves (6). Once again the computer indicated that solitary waves would emerge from collision having the same shapes and velocities with which they entered. Zabusky and Kruskal coined the term soliton to indicate this remarkable property.

Although the term soliton was originally applied only to solitary waves of the KdV equation, there are now several nonlinear wave equations known to exhibit similar effects, and the term is often used in wider context without formal definition.

A soliton ϕ_S (x-ut) is defined (4) as a solitary wave solution of a wave equation which assymptotically preserves its shape and velocity upon collision with another solitary wave. That is, given any solution $\phi(x,t)$ composed only of solitary waves for large negative time.

$$\phi(x,t) \sim \sum_{j=1}^{N} \phi_{ST}(\xi_j), \text{ as } t \to \infty \qquad (1.5)$$

$$\xi_j = x - u_j t \quad , \quad u_j \text{ constant} \qquad (1.6)$$

such solitary waves will be called solitons if they emerge from the interaction with no more than a phase shift, i.e.,

$$\phi(x,t) \sim \sum_{j=1}^{N} \phi_{ST}\,(\overline{\xi}_j) \quad , \quad \text{as } t \to +\infty \tag{1.7}$$

$$\overline{\xi}_j = x - u_j t + \delta_j \quad , \quad \delta_j \text{ constant} \tag{1.8}$$

For the sake of clarity the reader should realize the difference between solitary waves and shock waves. They are different in the sense that solitary waves occur in nonlinear dispersive media while shock waves occur in nonlinear dispersionless media. In nonlinear dispersionless media shock waves are formed because the pulse energy is continually injected into the higher frequency mode and the initial disturbance continues to steepen the region resulting eventually in shock waves (17). On the other hand, a balance is obtained between the effect of nonlinearity and that of dispersion in solitary waves (4). Some examples of naturally generated shock waves on earth are thunder, earthquakes, volcanic eruptions, and meteorite impacts. Some artificially generated shock waves are caused by bull whip, gunpowder, nuclear weapons, sonic boom, and atmospheric re-entry phenomena.

From our previous definition, the simplest example of a soliton is a pulse-like traveling wave solution of the dispersionless linear wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{1.9}$$

which is familiar to all electrical engineers. It may seem strange that a nonlinear and dispersive wave equation could even exhibit a solitary wave solution, much less a soliton. The effect of introducing dispersion without nonlinearity into Eq. (1.9) is to destroy the possibility of solitary waves because the various Fourier components of any initial conditions will propagate at different velocities. Introducing non-linearity without dispersion again removes the possibility for solitary waves because the pulse energy is continually injected into higher frequency modes. In the time domain this often appears as the formation of shock waves. But with both dispersion and nonlinearity, solitary waves can again form. The solitary wave can be qualitatively understood as representing balance between the effect of nonlinearity and that of dispersion.

1.3 Some Equations Exhibiting Solitary Wave Solutions

Solitary waves occur on propagating systems that are characterized by nonlinearity and dissipation. In this section we shall investigate some of these equations. Clearly this list of equations is incomplete; we expect it to grow in future years as solitons become better understood.

A. The Korteweg-deVries Equation

The equation (4)

$$\frac{\partial \phi}{\partial t} + \alpha\phi \frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0, \quad \alpha \text{ constant} \tag{1.10}$$

was first derived by Korteweg and deVries to describe the lossless propagation of shallow water waves (2). It is a useful approximation in many studies where one wishes to include a simple nonlinearity and a simple dispersive effect. Such studies include

1) magnetohydrodynamic waves in plasma (7)
2) pressure waves in liquid-gas bubble mixtures (8)
3) ion acoustic waves in plasma (9), (7)
4) the anharmonic lattice (10), (11)
5) rotating fluid down a tube (12).

In general, a rather large class of nearly hyperbolic mathematical systems has been shown to reduce to the KdV equation (13), (14).

Starting with the KdV equation (1.10) and assuming a traveling wave solution with velocity u as indicated by Eq. (1.1) to Eq. (1.4)

$$\frac{\partial\phi}{\partial t} = -u\frac{d\phi}{d\xi} \tag{1.11}$$

$$\frac{\partial\phi}{\partial x} = \frac{d\phi}{d\xi} \tag{1.12}$$

$$\frac{\partial^3\phi}{\partial x^3} = \frac{d^3\phi}{d\xi^3} \tag{1.13}$$

Substituting Eqs. (1.11), (1.12), and (1.13) into Eq. (1.10) yields

$$-u\frac{d\phi}{d\xi} + \alpha\phi\frac{d\phi}{d\xi} + \frac{d^3\phi}{d\xi^3} = 0$$

$$(\alpha\phi - u)\frac{d\phi}{d\xi} + \frac{d^3\phi}{d\xi^3} = 0 \tag{1.14}$$

where $\phi = \phi\,(x-ut) = \phi(\xi)$. Eq. (1.14) can be integrated to obtain

$$\frac{\partial^2\phi}{\partial\xi^2} = K_1 + u\phi - \frac{\alpha}{2}\phi^2 \tag{1.15}$$

where K_1 is a constant of the first integration. The second integration may be effected after multiplying both sides of Eq. (1.15) by $\frac{\partial\phi}{\partial\xi}$ then

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 = K_2 + K_1\phi + \frac{u}{2}\phi^2 - \frac{\alpha}{6}\phi^3$$

or

$$\left(\frac{\partial\phi}{\partial\xi}\right)^2 = 2K_2 + 2K_1\phi + u\phi^2 - \frac{\alpha}{3}\phi^3 \tag{1.16}$$

and the general traveling wave solution can be written in the form of elliptic integral (15)

$$\int_{\phi_o}^{\phi} \frac{d\phi}{\sqrt{P(\phi)}} = x - ut \tag{1.17}$$

where ϕ_o is the value of ϕ at $(x-ut) = 0$, and

$$P(\phi) \equiv 2K_2 + 2K_1\phi + u\phi^2 - \frac{\alpha}{3}\phi^3 \tag{1.18}$$

Since solitary waves are localized, their first and second derivatives must vanish as $\xi\to\pm\infty$. This condition together with Eq. (1.15), requires that $K_1 = 0$ and $K_2 = 0$. Then, Eq. (1.18) can be written as :

$$P(\phi) \equiv u\phi^2 - \frac{\alpha}{3}\phi^3 \tag{1.19}$$

Then Eq. (1.17) is readily integrated to yield (see Appendix A) a solitary wave

$$\phi\ (x-ut) = \frac{3u}{\alpha}\ \mathrm{sech}^2 \left[\sqrt{\frac{u}{2}}\ (x-ut) + \delta\right] \tag{1.20}$$

where

$$\delta = \mathrm{sech}^{-1} \sqrt{\frac{\alpha\phi_o}{3u}}$$

Some basic properties of KdV solitary waves can be seen from Eq.(1.17) and Eq. (1.20).

(1) The amplitude of a KdV solution increases with its velocity, and its width is inversely proportional to the square root of its velocity.

(2) The sign of a solitary wave solution of Eq. (1.10) depends upon the sign of α.

(3) The KdV-solitary waves are unidirectional, i.e., u cannot be negative for solitons of Eq. (1.10) since $\sqrt{P\ (\phi)}$ must remain real.

In 1965 Zabusky and Kruskal published numerical results indicating the formation of solitons (6). More recently, these computations have been extended by Tappert (16). One explicit analytic expression describing two interacting solitons is (17),

$$\phi = (-12)\ \frac{3 + 4\cosh\,(2x - 8t) + \cosh\,(4x - 64t)}{[3\cosh\,(x-28t) + \cosh\,(3x - 36t)]^2} \tag{1.21}$$

B. Nonlinear Lattice Equations

The electrical engineer is familiar with the lattice of mass points connected by springs as a mechanical analog for the low-pass electric filter.

The physicist is interested in such a system to model the propagation of sound waves through a crystal lattice. The "Toda lattice" equations (18), (19)

$$m \frac{d^2 r_n}{d t^2} = a \left(e^{-br_{n-1}} + e^{-br_{n+1}} - 2e^{-br_n} \right) \tag{1.22}$$

where r_n is the mutual displacement $r_n = y_n - y_{n-1}$, y_n being the displacement of the nth mass. a and b are positive constants, $K = ab$ the linear spring constant. The solution of the form (19)

$$e^{-br_{n-1}} = \sinh^2\kappa \operatorname{sech}^2 (\kappa n - \beta t) \tag{1.23}$$

with

$$\beta = \sqrt{\frac{ab}{m}} \sinh \kappa \tag{1.24}$$

represents a solitary wave pulse, which we may call the (lattice-) soliton.

C. The Nonlinear Schrödinger Equation (20)

$$\frac{\partial^2\phi}{\partial x^2} + i \frac{\partial\phi}{\partial t} + k |\phi|^2 \phi = 0 \tag{1.25}$$

can be derived from the Lagrangian density

$$L = \frac{i}{2} \left[\phi \left(\frac{\partial\phi}{\partial t}\right)^* - \phi^* \frac{\partial\phi}{\partial t} \right] + \left| \frac{\partial\phi}{\partial x} \right|^2 - \frac{k}{2} |\phi|^4 \tag{1.26}$$

where ϕ^* is the complex conjugate of ϕ. This equation has been used to describe (4)

(1) stationary two-dimensional self focusing of plasma waves,

(2) one-dimensional self modulation of a monochromatic wave,

(3) propagation of a heat pulse in a solid.

Envelope solitary wave solution of this equation is (4)

$$\phi = \phi_0 \operatorname{sech} \left[\sqrt{\frac{k}{2}}\ \phi_0 (x-u_e t)\right] . \exp\left[i \left(\frac{u_e}{2}\right)(x-u_c t)\right] \tag{1.27}$$

where u_e and u_c are the "envelope" and the carrier velocities respectively.

D. The sine-Gordon Equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi \tag{1.28}$$

The sine-Gordon equation has been used to describe (4)

(1) propagation of a crystal dislocation,

(2) Bloch motion of magnetic crystals,

(3) propagation of magnetic flux on a Josephson line.

Solitary wave solution of the sine-Gordon equation which corresponds to a rotation in ϕ by 2π (as x goes from $-\infty$ to $+\infty$) have the form

$$\phi = 4\tan^{-1}\left[\exp \pm \frac{x-ut}{\sqrt{1-u^2}}\right] \tag{1.29}$$

1.4 Experimental Observations

In 1970 Ikezi, Taylor, and Baker (21) gave experimental results on the formation and interaction of ion-acoustic solitons. The formation and propagation of ion-acoustic solitons were observed experimentally. The character of a solitary pulse was observed to follow the predictions of the KdV equation with respect to shape and velocity of the soliton. The interaction between two solitons was modified significantly by dissipation. However, the nonlinear nature of the interaction was confirmed for solitons moving in the same direction. Solitons moving in the opposite direction and colliding had very little effect on each other.

Later, in 1972 Hershkowitz, Romesse, and Montogemery (22) studied experimentally solitons in a double plasma device. They were able to observe that solitons retain their identity and the reduction in amplitude was accompanied by reductions in velocity and an increase in width.

In 1973 Ikezi (23) performed experiments on ion-acoustic solitary waves. He was able to verify that as the amplitude of the soliton is increased, the width decreases and the velocity increases.

CHAPTER 2

LATTICE SOLITONS IN NONLINEAR LUMPED NETWORKS

1.1 Introduction

Analytical solutions of the equation of motion in an anharmonic one-dimensional lattice as well as a proof of the existence of solitons was found by Toda (18). He also proved that the lattice solitons pass through one another without losing their identity (19). Encouraged by his results, a nonlinear network was constructed by Hirota (24) in order to illustrate some of the fundamental properties of solitons in terms of voltage solitons in a nonlinear LC network. This network is an equivalent system to one-dimensional nonlinear lattice and consists of a ladder-type LC circuit containing constant inductors and voltage dependent capacitors. We will adopt this circuit to study the fundamental properties of lattice solitons by using nonlinear lumped elements.

1.2 Network Solutions

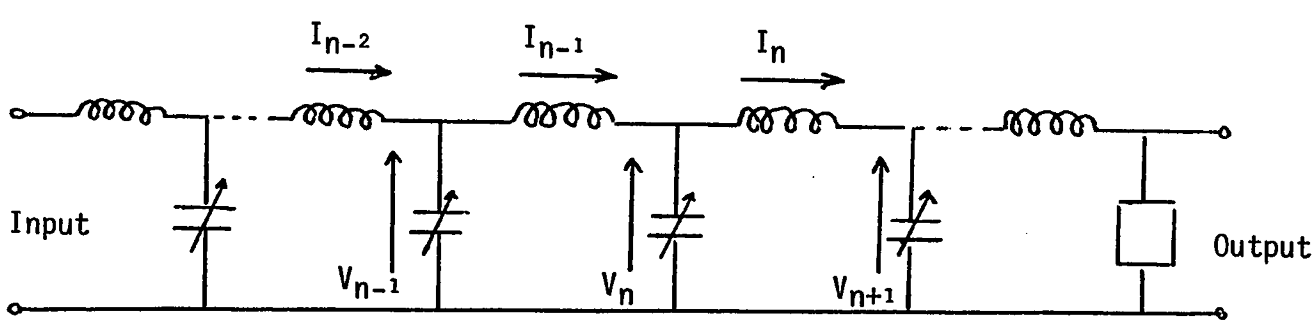

Fig. 2.1 The nonlinear network equivalent to a one-dimensional anharmonic lattice.

In this section we consider the solution of a cascade of four-terminal nonlinear LC sections as shown in Fig. 2.1. The propagation equations of the network are

$$\frac{\partial}{\partial t} L \ I_n (t) = V_n (t) - V_{n+1} (t) \qquad (2.1)$$

$$\frac{\partial}{\partial t} Q_n (t) = I_{n-1}(t) - I_n (t) \qquad (2.2)$$

with

$$Q_n (t) = C (V_n (t)) \ V_n (t) \qquad (2.3)$$

where $V_n (t)$ is the voltage present across the nth nonlinear capacitor with capacitance $C (V_n (t))$, and $I_n (t)$ is the current passing through the nth linear inductor with constant inductance L. The capacitance $C (V_n(t))$ is assumed to decrease with increasing voltage.

An analytical expression for a lattice soliton (18) can be obtained if one assumes that the capacitors whose capacitance changes with the signal voltage V_n are represented by (24)

$$C (V_n) = C_o (V_o/V_n) \ln (1+V_n/V_o) \qquad (2.4)$$

where C_o is a constant capacitance and V_o is a constant voltage applied to each of the capacitors. The above equation is representative of the behavior of hyperabrupt varactor diodes. Hirota (25) showed that the charge Q_n stored in the nth capacitor is given by

$$Q_n = C_o V_o \ln (1+V_n/V_o) \qquad (2.5)$$

From Eq. (2.2)

$$\frac{\partial^2}{\partial t^2} Q_n (t) = \frac{\partial}{\partial t} I_{n-1} (t) - \frac{\partial}{\partial t} I_n (t) \tag{2.6}$$

From Eq. (2.1)

$$\frac{\partial}{\partial} I_n (t) = \frac{1}{L} (V_n (t) - V_{n+1} (t)) \tag{2.7}$$

and

$$\frac{\partial}{\partial t} I_{n-1} (t) = \frac{1}{L} (V_{n-1} (t) - V_n (t)) \tag{2.8}$$

Substituting Eqs. (2.7) and (2.8) into Eq. (2.6) gives

$$\frac{\partial^2}{\partial t^2} Q_n (t) = \frac{1}{L} (V_{n-1} (t) - V_n (t) - \frac{1}{L} (V_n(t)-V_{n+1}(t))$$

$$L \frac{\partial^2}{\partial t^2} Q_n (t) = V_{n+1} (t) + V_{n-1} (t) - 2 V_n (t) \tag{2.9}$$

Substituting Eq. (2.5) into Eq. (2.9) yields

$$L \frac{\partial^2}{\partial t^2} [C_o V_o \ln (1 + V_n/V_o)] = V_{n+1} + V_{n-1} - 2 V_n$$

or,

$$L C_o \frac{\partial^2}{\partial t^2} \ln (1 + V_n/V_o) = V_{n+1}/V_o + V_{n-1}/V_o - 2V_n/V_o \tag{2.10}$$

If we assume that

$$1 + V_n/V_o = e^{br_n} \tag{2.11}$$

and

$$m/ab = L\,C_o \tag{2.12}$$

equation (2.10) reduces to the Toda equation (18)

$$m\,\ddot{r}_n = a\,(e^{br_{n-1}} + e^{br_{n+1}} - 2\,e^{br_n}) \tag{2.13}$$

In the above equation a and b are positive real constants and ab is the linear spring constant while m is the vibrating mass and r_n is the displacement of the nth mass and the (n-1)th mass in the vibrating system studied by Toda. Eq. (2.13) has a solution of the form

$$e^{br_n} - 1 = \sinh^2\kappa\,\operatorname{sech}^2(\kappa n - \beta t) \tag{2.14}$$

where

$$\beta = \sqrt{ab/m}\,\sinh\kappa \tag{2.15}$$

This solution is analogous to the solution of Eq. (2.8) with

$$\kappa = P \tag{2.16}$$

$$\sinh\kappa = \sinh P = \Omega \tag{2.17}$$

where Ω^2 is the maximum amplitude of the soliton and

$$\beta = \frac{\sinh P}{\sqrt{L C_o}} = \frac{\Omega}{\sqrt{L C_o}} \tag{2.18}$$

By comparison we deduce that the solution of Eq. (2.10) is

$$\frac{V_n(t)}{V_o} = \Omega^2 \operatorname{sech}^2 (Pn - \Omega t / \sqrt{L C_o}) \tag{2.19}$$

If we let $\tau = t / \sqrt{L C_o}$, then

$$\frac{V_n(\tau)}{V_o} = \Omega^2 \operatorname{sech}^2 (Pn - \Omega\tau) \tag{2.20}$$

or

$$\frac{V_n(\tau)}{V_o} = \Omega^2 \operatorname{sech}^2 (\Omega\tau - Pn) \tag{2.21}$$

The normalized velocity of the soliton is given by

$$v_o = \Omega / P \tag{2.22}$$

$$= (\sinh P) / P \tag{2.23}$$

It should be noted that the solitary-wave pulse solution of Eq. (2.21) vanishes in the limit $\Omega \to 0$, which implies that the solution cannot be obtained by using the usual perturbation theory.

It is also worth observing that workers investigating soliton behavior in different areas of physics have developed equations similar to that of Eq. (2.21). Ikezi et al (21) have developed the equation of the electron density profile in ion-acoustic plasma in the form

$$\tilde{n} = \delta n \ \operatorname{sech}^2 \left(\frac{x - ut}{D} \right) \tag{2.24}$$

$$u = v_s \left(1 + \frac{1}{3} \delta n/n_0\right)$$

where D is the width of the soliton and δn is the amplitude of the soliton. Also, in his study of motion in a one-dimensional mass point lattice with nonlinear potential, Toda found that the force between the mass points was a solitary wave having the following form

$$\phi = \frac{1}{4} \frac{m}{ab} \beta^2 \operatorname{sech}^2 \left[\frac{1}{2} (\kappa n - \beta t) + \delta \right] \tag{2.25}$$

The implication that solitary wave phenomena occur in different areas of physical science is thus made clear.

1.3. Soliton-Soliton Interaction

An analytical expression for the soliton-soliton interaction can be found for the special capacitance expressed by Eq. (2.4). Hirota (24) has found this expression, and we rewrite his solution and express the voltage $\frac{V_n(t)}{V_0}$ in the following form :

$$\frac{V_n(\tau)}{V_0} = \frac{\Omega_1^2 \operatorname{sech}^2 \eta_1 + \Omega_2^2 \operatorname{sech}^2 \eta_2 + A_0 \operatorname{sech}^2 \eta_1 \operatorname{sech}^2 \eta_2}{[\cosh(\phi/2) + \sinh(\phi/2) \tanh \eta_1 \tanh \eta_2]^2} \tag{2.26}$$

where

$$\eta_i = \Omega_i \tau - P_i n - \eta_i^{\circ} \tag{2.27}$$

$$\Omega_i^2 = \sinh^2 P_i \quad , \quad \text{for } i = 1,2 \tag{2.28}$$

$$\exp(2\phi) = -\left[(\Omega_1 - \Omega_2)^2 - \sinh^2(P_1 - P_2)\right] / \left[(\Omega_1 + \Omega_2)^2 - \sinh^2(P_1 + P_2)\right] \tag{2.29}$$

and

$$A_0 = \sinh(\phi/2)\left[(\Omega_1^2 + \Omega_2^2)\sinh(\phi/2) + 2\,\Omega_1\,\Omega_2\cosh(\phi/2)\right] \tag{2.30}$$

where P_i is a finite real constant which determines the amplitude of the ith soliton, and η_i° is a finite real constant which relates to the phase of the ith lattice solitons.

In the cases where two solitons move in the same direction, we have

$$\Omega_1 = \sinh P_1 \tag{2.31}$$

$$\Omega_2 = \sinh P_2 \tag{2.32}$$

and

$$\exp(2\phi) = \sinh^2\left[(P_1 - P_2)/2\right] / \sinh^2\left[(P_1 + P_2)/2\right]$$

The expression for $\dfrac{V_n(\tau)}{V_0}$ is the same as in Eq. (2.26).

In the cases where two solitons move in opposite directions, we have

$$\Omega_1 = \sinh P_1 \tag{2.34}$$

$$\Omega_2 = \sinh P_2 \tag{2.35}$$

and

$$\exp(2\phi) = \cosh^2 [(P_1 - P_2)/2] / \cosh^2 [(P_1 + P_2)/2] \tag{2.36}$$

The expression for $\frac{V_n(\tau)}{V_o}$ is the same as in Eq. (2.26).

CHAPTER 3

SOLITONS PROPERTIES

3.1 Introduction

The aim of this chapter is to verify numerically that :

(a) A single voltage soliton travels in the lumped nonlinear network of Fig. (2.1) without changing either shape or velocity.

(b) The amplitude of a single voltage soliton depends on the velocity of propagation.

(c) Two voltage solitons moving in opposite directions collide and emerge after collision without changing shapes.

(d) Two voltage solitons moving in the same direction with different velocities collide and emerge after collision without changing shapes.

(e) The integral of the voltage with respect to the normalized time is constant for two voltage solitons moving in opposite directions and for two voltage solitons moving in the same direction.

(f) A single current soliton travels without changing shape or velocity.

(g) Two current solitons moving in opposite directions collide and emerge after collision without changing shapes.

(h) Two current solitons moving in the same direction with different velocities collide and emerge after collision without changing shapes.

(i) The integral of the power with respect to the normalized time for two solitons moving in opposite directions and for two solitons moving in the same direction is constant and independent of the position n.

3.2 Retaining of Shape of a Single Voltage Soliton

The motion of a single soliton is described by Eq. (2.21),

$$\frac{V_n(\tau)}{V_0} = \Omega^2 \ \mathrm{sech}^2 \ (\Omega\tau - Pn) \tag{3.1}$$

where τ is the normalized time given by

$$\tau = t \ /\sqrt{L C_0} \tag{3.2}$$

Ω^2 is the amplitude of a single voltage soliton given by,

$$\Omega = \sinh \ P \tag{3.3}$$

where P is a finite real constant.

A computer program was written (see Appendix B) to compute the normalized voltage soliton (with Ω^2 =13.15 and P=1.999849137) as a function of τ for n=-10 to n=10. The results for n=-10,-6,6, and 10 are tabulated in Table (3.6). These results are plotted as shown in Fig. (3.1) to (3.4).

The first observation regarding this voltage soliton is that it travels from left to right. The second observation is that it has the same maximum amplitude of 13.15 and width of 2.0 for n=-10,6,6, and 10. The third observation is the retaining of shape for the mentioned values of n.

τ	$V_{-10}(\tau)/V_0$	τ	$V_{-6}(\tau)/V_0$	τ	$V_6(\tau)/V_0$	τ	$V_{10}(\tau)/V_0$
-6.5	0.0414	-4.3	0.0397	2.3	0.0349	4.5	0.0334
-6.4	0.0854	-4.2	0.0818	2.4	0.0719	4.6	0.0689
-6.3	0.1758	-4.1	0.1685	2.5	0.1482	4.7	0.1419
-6.2	0.3606	-4.0	0.3456	2.6	0.3042	4.8	0.2914
-6.1	0.7338	-3.9	0.7038	2.7	0.6204	4.9	0.5948
-6.0	1.4709	-3.8	1.4123	2.8	1.2494	5.0	1.1990
-5.9	2.8597	-3.7	2.7526	2.9	2.4515	5.1	2.3576
-5.8	5.2418	-3.6	5.0688	3.0	4.5730	5.2	4.4155
-5.7	8.6368	-3.5	8.4191	3.1	7.7704	5.3	7.5565
-5.6	11.9720	-3.4	11.8140	3.2	11.2950	5.4	11.1090
-5.5	13.1120	-3.3	13.1360	3.3	13.1360	5.5	13.1120
-5.4	11.1080	-3.2	11.2940	3.4	11.8130	5.6	11.9720
-5.3	7.5556	-3.1	7.7686	3.5	8.4175	5.7	8.6359
-5.2	4.4147	-3.0	4.5717	3.6	5.0676	5.8	5.2412
-5.1	2.3571	-2.9	2.4507	3.7	2.7518	5.9	2.8593
-5.0	1.1988	-2.8	1.2489	3.8	1.4119	6.0	1.4707
-4.9	0.5947	-2.7	0.6202	3.9	0.7036	6.1	0.7338
-4.8	0.2914	-2.6	0.3040	4.0	0.3455	6.2	0.3605
-4.7	0.1419	-2.5	0.1481	4.1	0.1684	6.3	0.1758
-4.6	0.0689	-2.4	0.0719	4.2	0.0818	6.4	0.0854
-4.5	0.0334	-2.3	0.0349	4.3	0.0397	6.5	0.0414

Table 3.1 Theoretical results expressing one voltage soliton for n=-10,-6,6,10

(Ω^2 =13.15)

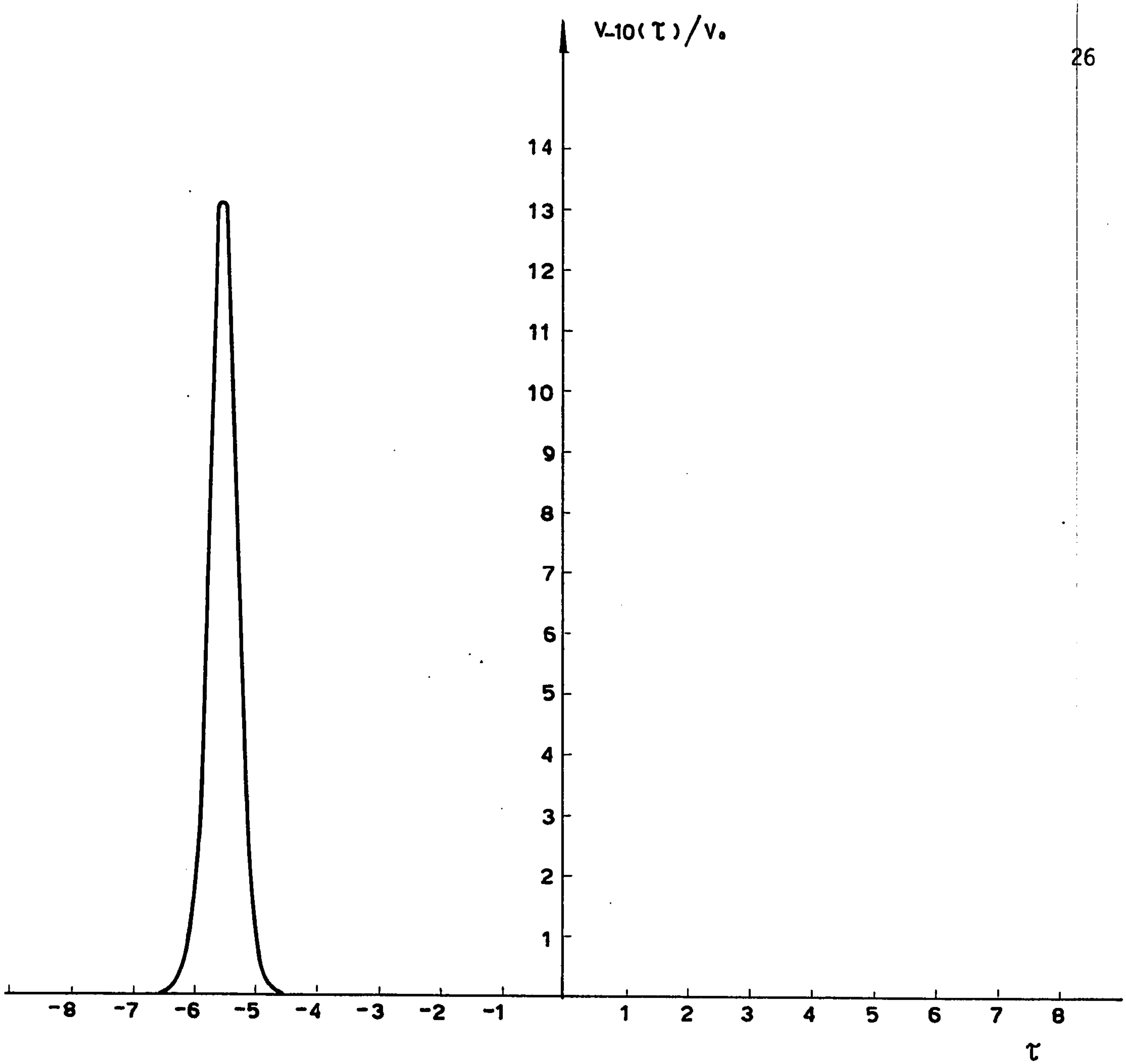

FIG. 3·1 NORMALIZED VOLTAGE SOLITON FOR $n = -10$ ($\Omega^2 = 13 \cdot 15$)

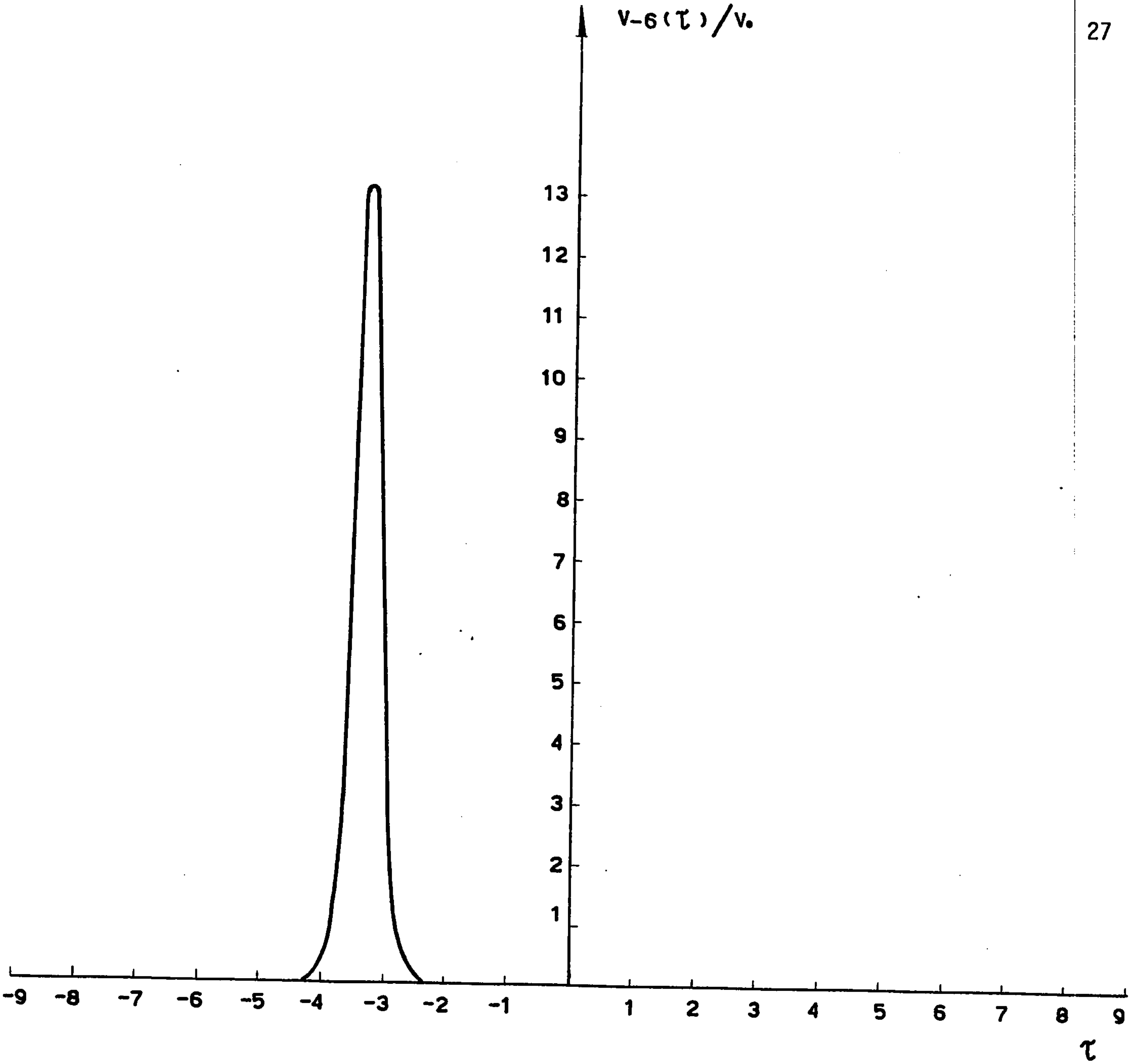

FIG. 3·2 NORMALIZED VOLTAGE SOLITON FOR $n = -6$ ($\Omega^2 = 13{\cdot}15$)

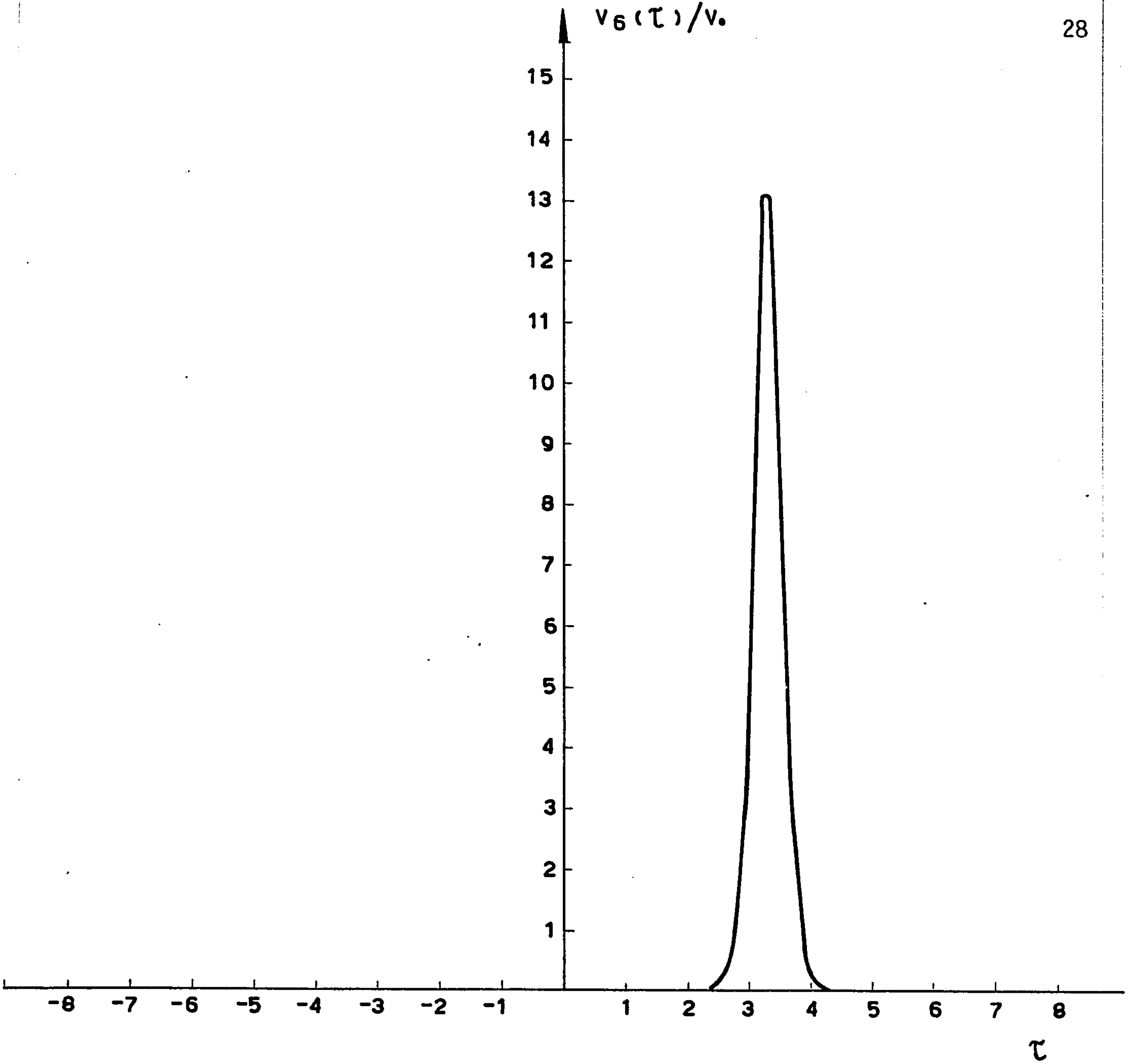

FIG. 3.3 NORMALIZED VOLTAGE SOLITON FOR $n = 6$ ($\Omega^2 = 13.15$)

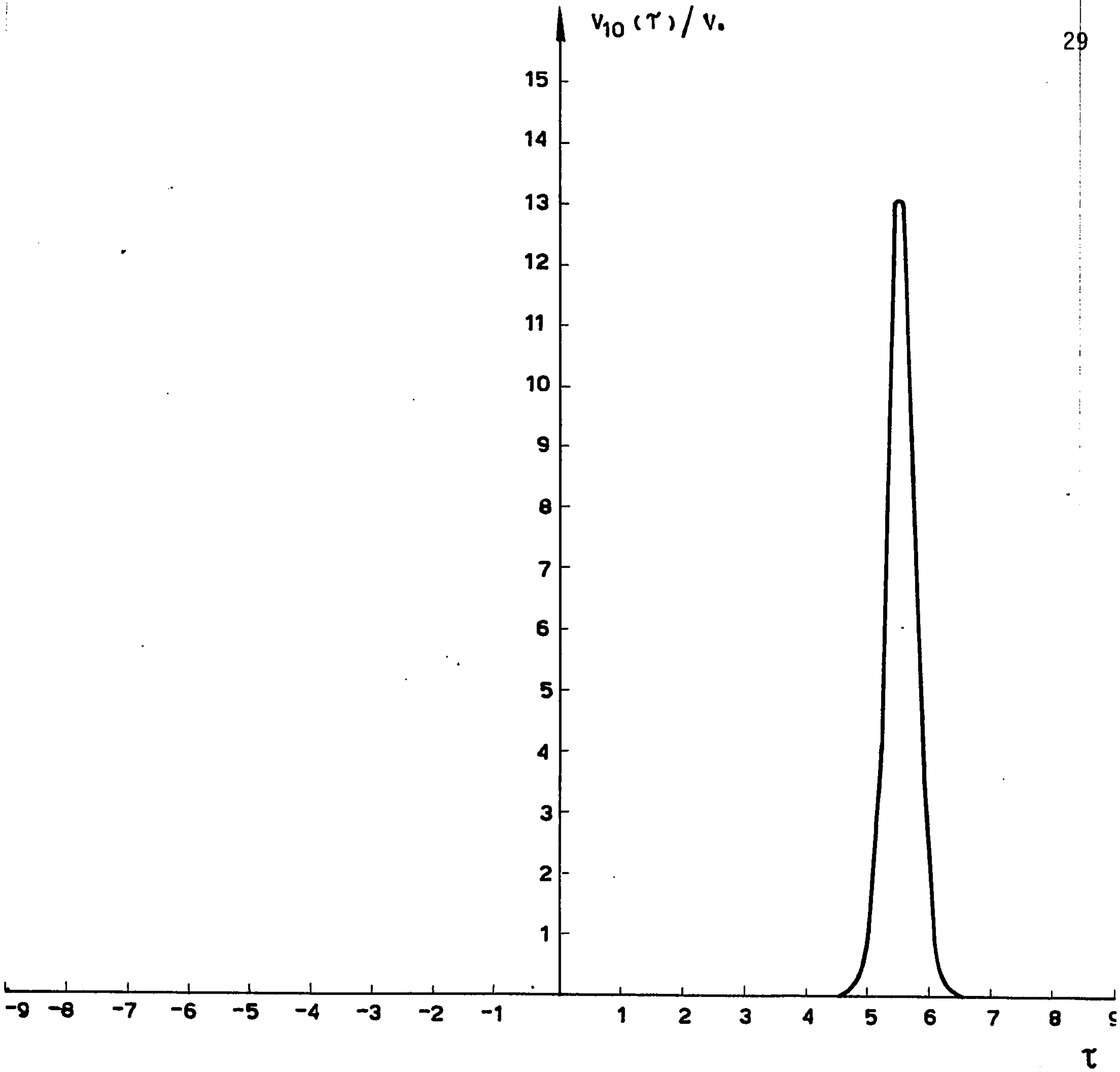

FIG. 3· 4 NORMALIZED VOLTAGE SOLITON FOR n = 10 (Ω^2 = 13·15)

3.3 Dependence of the Amplitude on the Velocity of Propagation

The normalized velocity of propagation is expressed by Eq. (2.22) as,

$$V_0 = \Omega / P \tag{3.4}$$

and the actual velocity is

$$v = v_0 \; / \sqrt{L\, C_0}$$

A computer program was written (see Appendix C) to compute the magnitude of a single voltage soliton as a function of τ for three different velocities of 1.8133, 1.4845, and 1.3061. The results for n=-5,0, and 5 are tabulated in Table (3.2), (3.3), and (3.4) for the three different velocities. The results of these computations are plotted in Fig. (3.5), (3.6), and (3.7). As seen in the graphs, the faster soliton has a larger amplitude than the slower ones. Also the width of the faster soliton is smaller than the width of the slower solitons.

τ	$V_{-5}(\tau)/V_0$ (v_0=1.8133)	$V_{-5}(\tau)/V_0$ (v_0=1.4845)	$V_{-5}(\tau)/V_0$ (v_0=1.3061)
-5.0	0.0000	0.0097	0.2080
-4.9	0.0000	0.0156	0.2877
-4.8	0.0000	0.2050	0.3958
-4.7	0.0000	0.0402	0.5400
-4.6	0.0001	0.0645	0.7289
-4.5	0.0002	0.1034	0.9696
-4.4	0.0004	0.1653	0.1265
-4.3	0.0007	0.2634	1.6086
-4.2	0.0015	0.4175	1.9809
-4.1	0.0031	0.6562	2.3447
-4.0	0.0064	1.0177	2.6484
-3.9	0.0132	1.5461	2.8376
-3.8	0.0273	2.2766	2.8734
-3.7	0.0564	3.2020	2.7479
-3.6	0.1162	4.2234	2.4879
-3.5	0.2388	5.1205	2.1435
-3.4	0.4885	5.6076	1.7693
-3.3	0.9891	5.4953	1.4096
-3.2	1.9617	4.8292	1.0918
-3.1	3.7363	3.8551	0.8273
-3.0	6.5915	2.8493	0.6165
-2.9	10.1790	1.9887	0.4538
-2.8	12.8420	1.3335	0.3310
-2.7	12.5960	0.8703	0.2398
-2.6	9.6483	0.5581	0.1729
-2.5	6.0992	0.3538	0.1242
-2.4	3.4076	0.2227	0.0890
-2.3	1.7752	0.1396	0.0637
-2.2	0.8914	0.0872	0.0455
-2.1	0.4394	0.0544	0.0325
-2.0	0.2146	0.0339	0.0232
-1.9	0.1044	0.0211	0.0165
-1.8	0.0506	0.0131	0.0118
-1.7	0.0245	0.0082	0.0084

Table 3.2 Dependence of the amplitude of a single voltage soliton on the velocity of propagation for n=-5

τ	$V_0(\tau)/V_0$ (V_0=1.8133)	$V_0(\tau)/V_0$ (V_0=1.4845)	$V_0(\tau)/V_0$ (V_0=1.3061)
-2.0	0.0000	0.0130	0.0017
-1.9	0.0000	0.0182	0.0027
-1.8	0.0001	0.0255	0.0044
-1.7	0.0002	0.0357	0.0070
-1.6	0.0005	0.0500	0.0113
-1.5	0.0010	0.0700	0.0181
-1.4	0.0020	0.0978	0.0291
-1.3	0.0042	0.1364	0.0468
-1.2	0.0087	0.1900	0.0750
-1.1	0.0180	0.2627	0.1201
-1.0	0.0372	0.3620	0.1919
-0.9	0.0767	0.4953	0.3054
-0.8	0.1580	0.6708	0.4830
-0.7	0.3242	0.8963	0.7563
-0.6	0.6608	1.1763	1.1664
-0.5	1.3284	1.5077	1.7568
-0.4	2.5981	1.8750	2.5537
-0.3	4.8162	2.2459	3.5262
-0.2	8.0931	2.5722	4.5372
-0.1	11.5620	2.7986	5.3336
0.0	13.1500	2.8800	5.6400
0.1	11.5610	2.7986	5.3334
0.2	8.0921	2.5721	4.5370
0.3	4.8153	2.2458	3.5259
0.4	2.5975	1.8749	2.5534
0.5	1.3281	1.5076	1.7565
0.6	0.6606	1.1762	1.1662
0.7	0.3241	0.8962	0.7562
0.8	0.1580	0.6707	0.4829
0.9	0.0767	0.4952	0.3054
1.0	0.0372	0.3620	0.1919
1.1	0.0180	0.2627	0.1201
1.2	0.0087	0.1896	0.0750
1.3	0.0042	0.1364	0.0468
1.4	0.0020	0.0978	0.0291
1.5	0.0010	0.0700	0.0181

Table 3.3 Dependence of the amplitude of a single voltage soliton on the velocity of propagation for n = 0

τ	$V_5(\tau)/V_0$ (v_0=1.8133)	$V_5(\tau)/V_0$ (v_0=1.4845)	$V_5(\tau)/V_0$ (v_0=1.3061)
1.7	0.0245	0.0082	0.0084
1.8	0.0506	0.0131	0.0118
1.9	0.1044	0.0211	0.0165
2.0	0.2146	0.0339	0.0231
2.1	0.4395	0.0544	0.0325
2.2	0.8915	0.0872	0.0455
2.3	1.7754	0.1396	0.0637
2.4	3.4080	0.2227	0.0890
2.5	6.0997	0.3538	0.1242
2.6	9.6488	0.5581	0.1729
2.7	12.5960	0.8704	0.2398
2.8	12.8410	1.3335	0.3310
2.9	10.1780	1.9888	0.4538
3.0	6.5911	2.8494	0.6165
3.1	3.3760	3.8552	0.8273
3.2	1.9616	4.8293	1.0918
3.3	0.9890	5.4953	1.4097
3.4	0.4885	5.6076	1.7693
3.5	0.2388	5.1204	2.1435
3.6	0.1162	4.2233	2.4879
3.7	0.0564	3.2019	2.7479
3.8	0.0273	2.2766	2.8734
3.9	0.0132	1.5461	2.8376
4.0	0.0064	1.0177	2.6484
4.1	0.0031	0.6562	2.3447
4.2	0.0015	0.4175	1.9809
4.3	0.0007	0.2634	1.6086
4.4	0.0004	0.1653	1.2648
4.5	0.0002	0.1034	0.9696
4.6	0.0001	0.0645	0.7290
4.7	0.0000	0.0402	0.5400
4.8	0.0000	0.0250	0.3958
4.9	0.0000	0.0156	0.2877
5.0	0.0000	0.0097	0.2080

Table 3.4 Dependence of the amplitude of a single voltage soliton on the velocity of propagation for n = 5

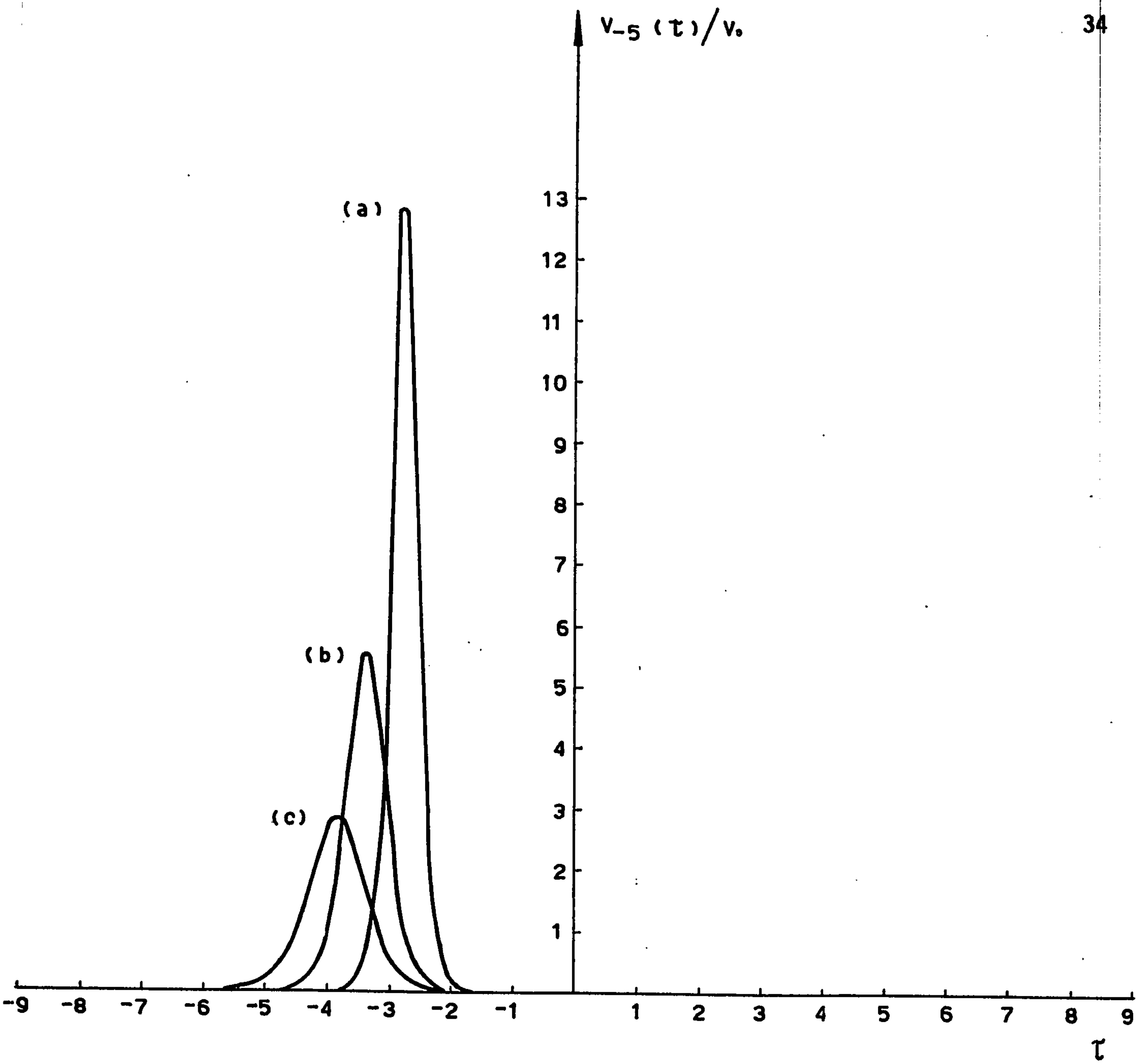

FIG. 3.5 DEPENDENCE OF THE AMPLITUDE OF A SINGLE SOLITON ON THE VELOCITY OF PROPAGATION, (a) $\Gamma_0 = 1.8133$, (b) $\Gamma_0 = 1.4845$, (c) $\Gamma_0 = 1.3061$

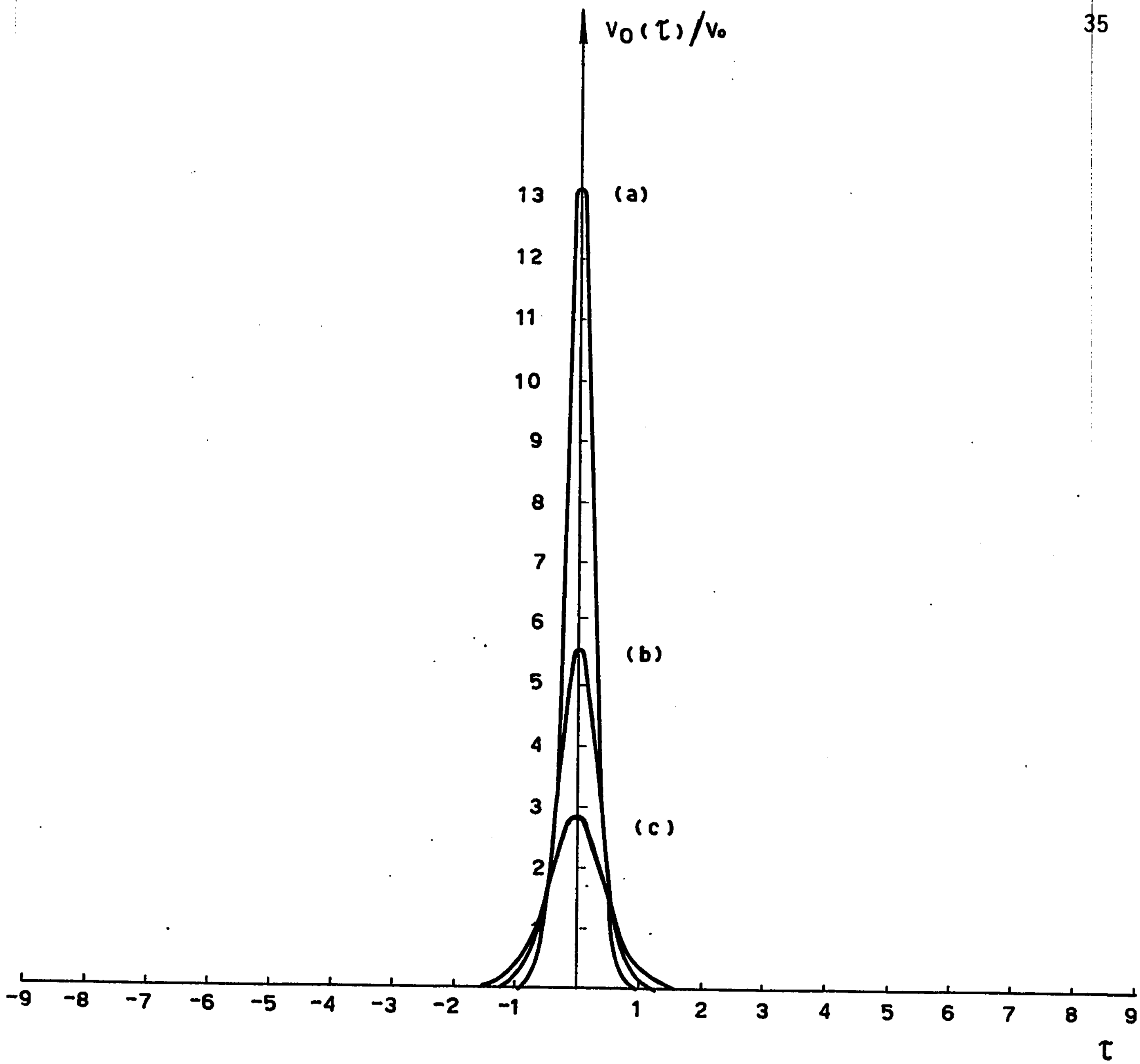

FIG. 3.6 DEPENDENCE OF THE AMPLITUDE OF A SINGLE SOLITON ON THE VELOCITY OF PROPAGATION, (a) $\sqrt{\sigma} = 1.8133$, (b) $\sqrt{\sigma} = 1.4845$, (c) $\sqrt{\sigma} = 1.3061$

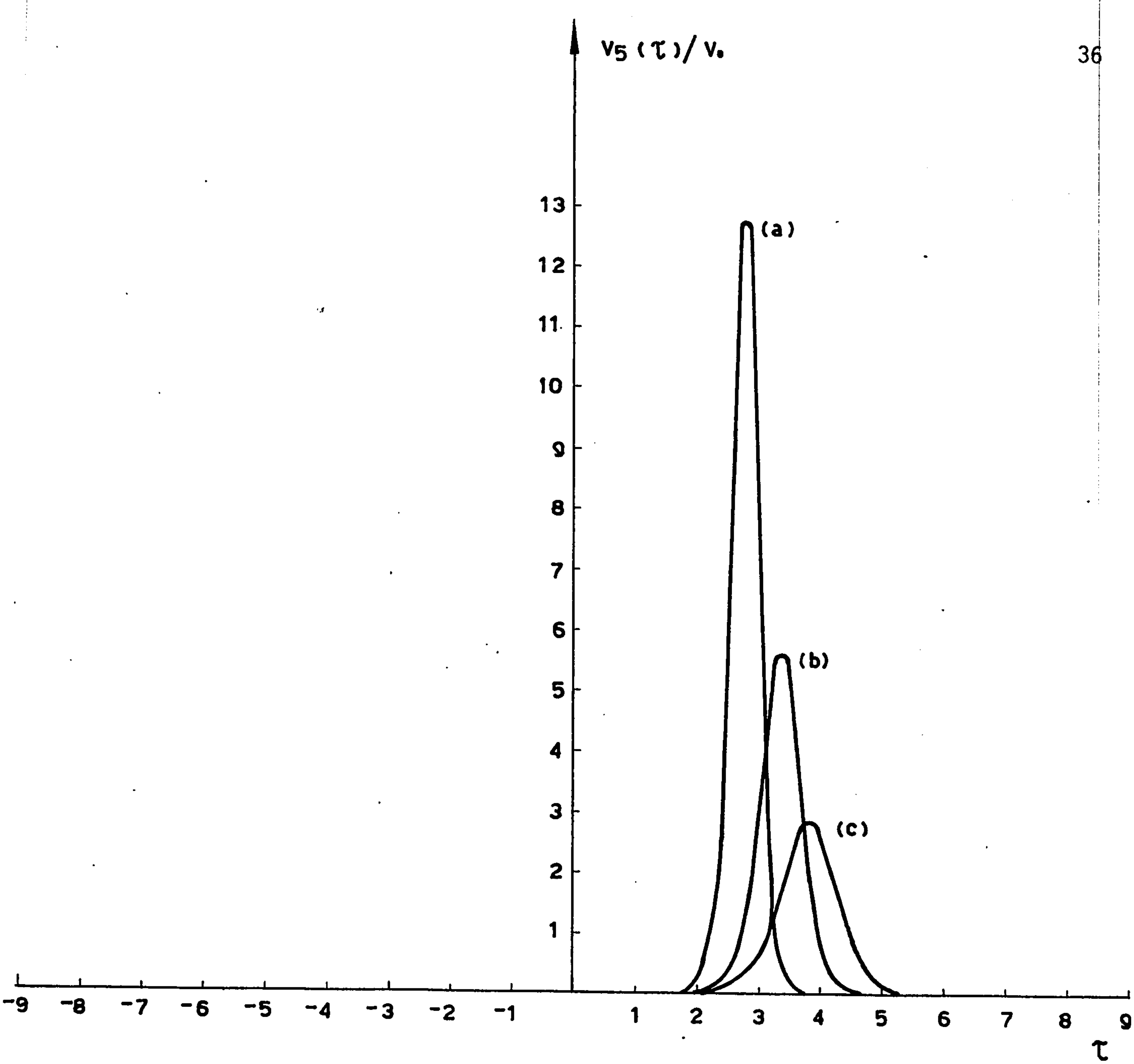

FIG. 3.7 DEPENDENCE OF THE AMPLITUDE OF A SINGLE SOLITON ON THE VELOCITY OF PROPAGATION,

(a) $V_0 = 1.8133$, (b) $V_0 = 1.4845$, (c) $V_0 = 1.3061$

3.4 Collision of Two Voltage Solitons Moving in Opposite Directions

The motion of two voltage solitons moving in opposite directions is expressed by Eq. (2.26), (2.27), (2.28), (2.34), (2.35), and (2.36) with $\Omega_1^2 = \Omega_2^2 = 13.15$.

A computer program was written (see Appendix D) to carry on the computations. The result of these computations are tabulated in Table (3.5) for n=-3,-1,0,1, and 3. These results are plotted in Fig. (3.8) to (3.12). The plots show that the two voltage solitons collide and emerge after collision without changing their shapes. At the interaction the joint maximum amplitude is not simply the sum of the two individual maximum of the two voltage solitons.

τ	$V_{-3}(\tau)/V_0$	τ	$V_{-1}(\tau)/V_0$	τ	$V_0(\tau)/V_0$	τ	$V_1(\tau)/V_0$	τ	$V_3(\tau)/V_0$
-2.4	0.0626	-1.3	0.0613	-0.9	0.0413	-1.3	0.0613	-2.4	0.0626
-2.3	0.1289	-1.2	0.1262	-0.8	0.0862	-1.2	0.1262	-2.3	0.1289
-2.2	0.2648	-1.1	0.2594	-0.7	0.1815	-1.1	0.2594	-2.2	0.2648
-2.1	0.5411	-1.0	0.5301	-0.6	0.3900	-1.0	0.5301	-2.1	0.5411
-2.0	1.0932	-0.9	1.0716	-0.5	0.8681	-0.9	1.0716	-2.0	1.0932
-1.9	2.1589	-0.8	2.1192	-0.4	2.0442	-0.8	2.1192	-1.9	2.1589
-1.8	4.0777	-0.7	4.0131	-0.3	5.1474	-0.7	4.0131	-1.8	4.0777
-1.7	7.0853	-0.6	7.0076	-0.2	13.2863	-0.6	7.0076	-1.7	7.0853
-1.6	10.6730	-0.5	10.6484	-0.1	29.6795	-0.5	10.6484	-1.6	10.6730
-1.5	13.0136	-0.4	13.1757	0.0	41.5534	-0.4	13.1757	-1.5	13.0136
-1.4	12.2970	-0.3	12.7837	0.1	29.6666	-0.3	12.7837	-1.4	12.2970
-1.3	9.1259	-0.2	10.0828	0.2	13.2779	-0.2	10.0828	-1.3	9.1249
-1.2	5.6439	-0.1	7.4063	0.3	5.1440	-0.1	7.4063	-1.2	5.6439
-1.1	3.1129	0.0	6.3700	0.4	2.0430	-0.0	6.3700	-1.1	3.1129
-1.0	1.6105	0.1	7.4077	0.5	0.8677	0.1	7.4077	-1.0	1.6105
-0.9	0.8059	0.2	10.0849	0.6	0.3898	0.2	10.0849	-0.9	0.8059
-0.8	0.3966	0.3	12.7850	0.7	0.1814	0.3	12.7850	-0.8	0.3966
-0.7	0.1935	0.4	13.1749	0.8	0.0861	0.4	13.1749	-0.7	0.1935
-0.6	0.0941	0.5	10.6460	0.9	0.0413	0.5	10.6460	-0.6	0.0941
-0.5	0.0457	0.6	7.0052			0.6	7.0052	-0.5	0.0457
0.5	0.0457	0.7	4.0115			0.7	4.0115	0.5	0.0457
0.6	0.0941	0.8	2.1182			0.8	2.1182	0.6	0.0941
0.7	0.1936	0.9	1.0711			0.9	1.0711	0.7	0.1936
0.8	0.3967	1.0	0.5298			1.0	0.5298	0.8	0.3967
0.9	0.8063	1.1	0.2592			1.1	0.2592	0.9	0.8063
1.0	1.6113	1.2	0.1262			1.2	0.1262	1.0	1.6113
1.1	3.1143	1.3	0.0612			1.3	0.0612	1.1	3.1143
1.2	5.6459							1.2	5.6459
1.3	9.1273							1.3	9.1273
1.4	12.2984							1.4	12.2984
1.5	13.0130							1.5	13.0130
1.6	10.6708							1.6	10.6708
1.7	7.0832							1.7	7.0832
1.8	4.0762							1.8	4.0762
1.9	2.1581							1.9	2.1581
2.0	1.0927							2.0	1.0927
2.1	0.5409							2.1	0.5409
2.2	0.2647							2.2	0.2647
2.3	0.1289							2.3	0.1289
2.4	0.0625							2.4	0.0625

Table 3.5 Theoretical results expressing two voltage solitons moving in opposite directions for n=-3,-1,0,1,3; ($\Omega_1^2 = \Omega_2^2 = 13.15$)

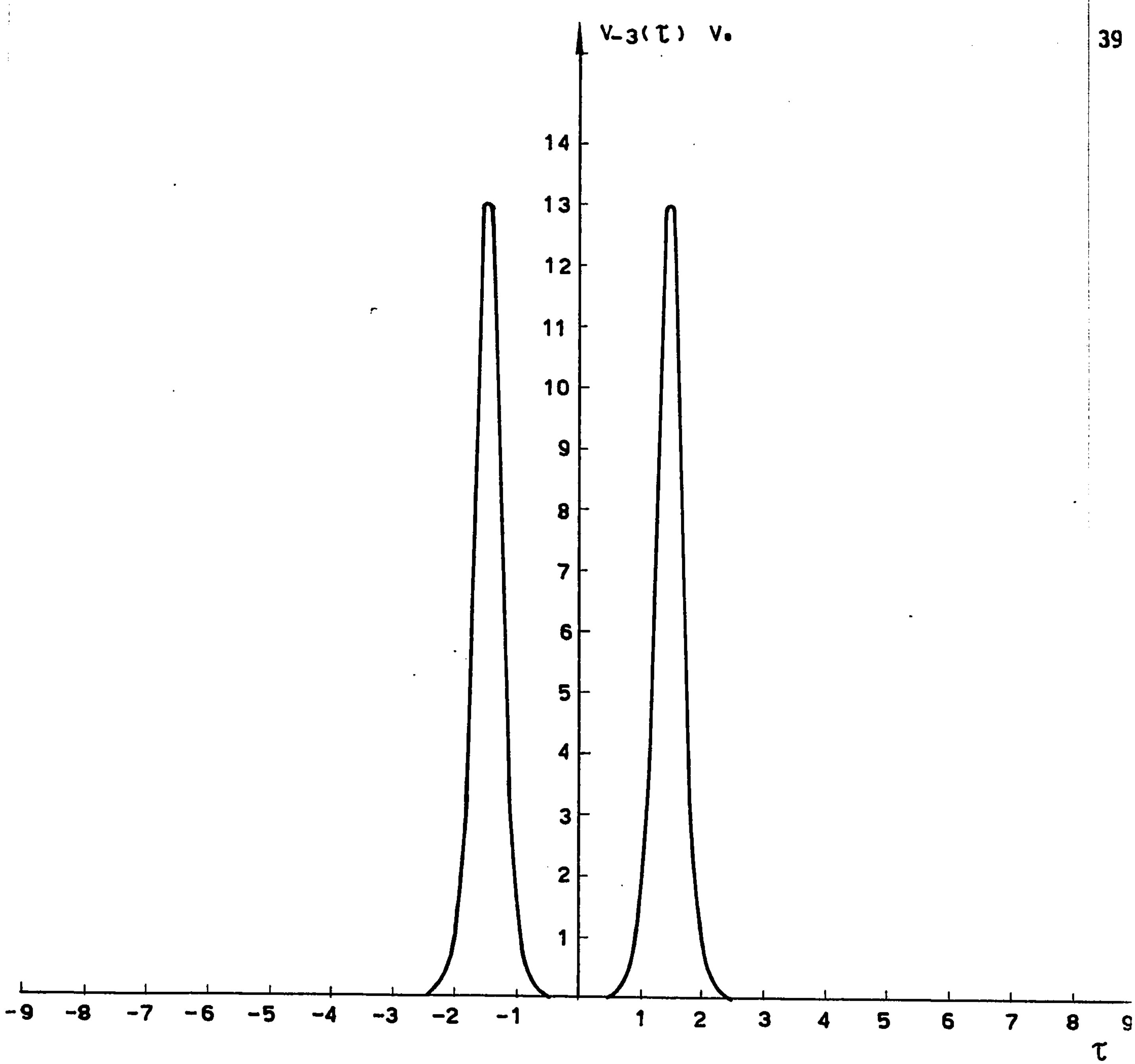

FIG. 3. 8 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = -3$ ($\Omega_1^2 = \Omega_2^2 = 13{\cdot}15$)

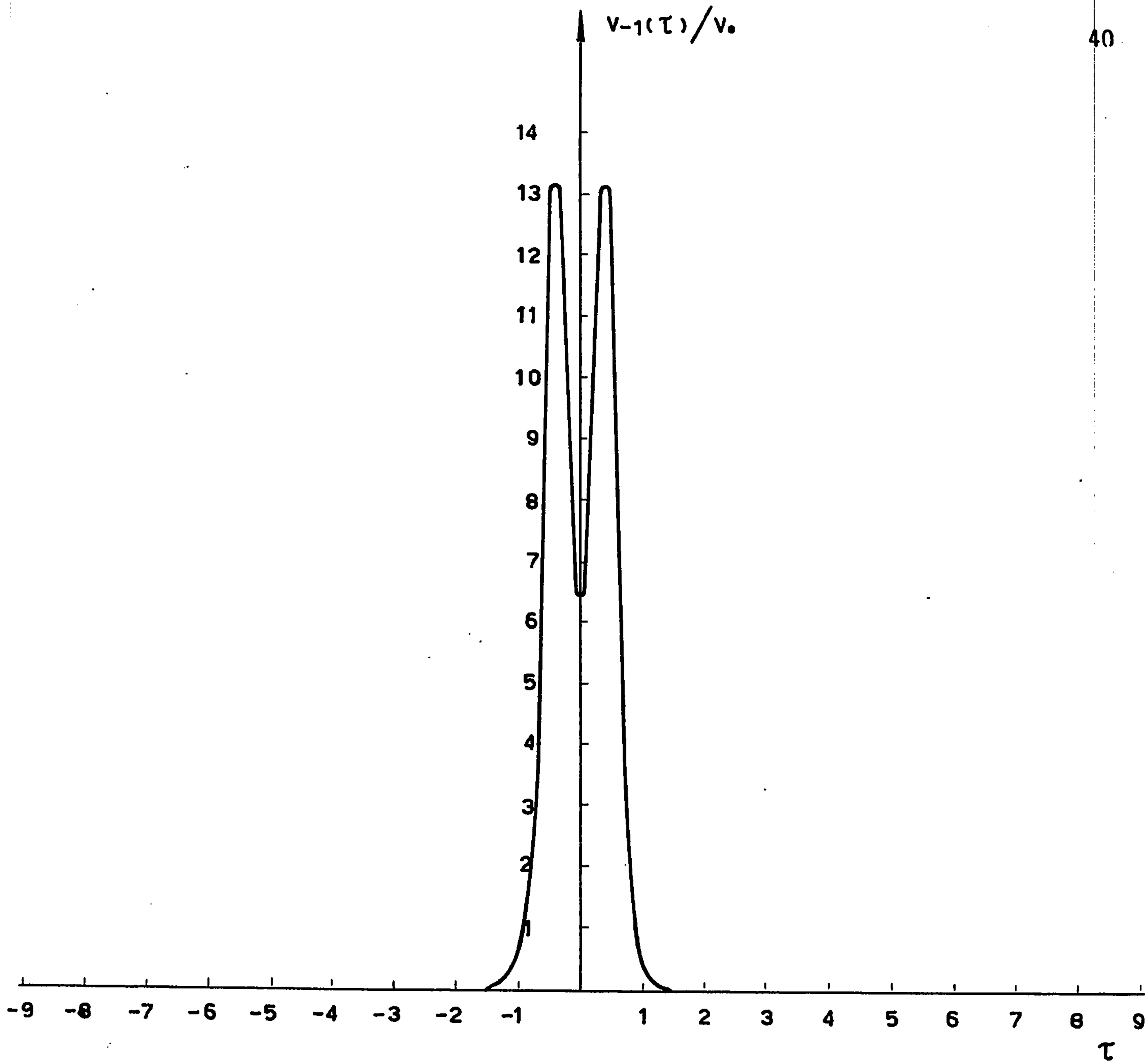

FIG. 3.9 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n=-1$ $(\Omega_1^2 = \Omega_2^2 = 13.15)$

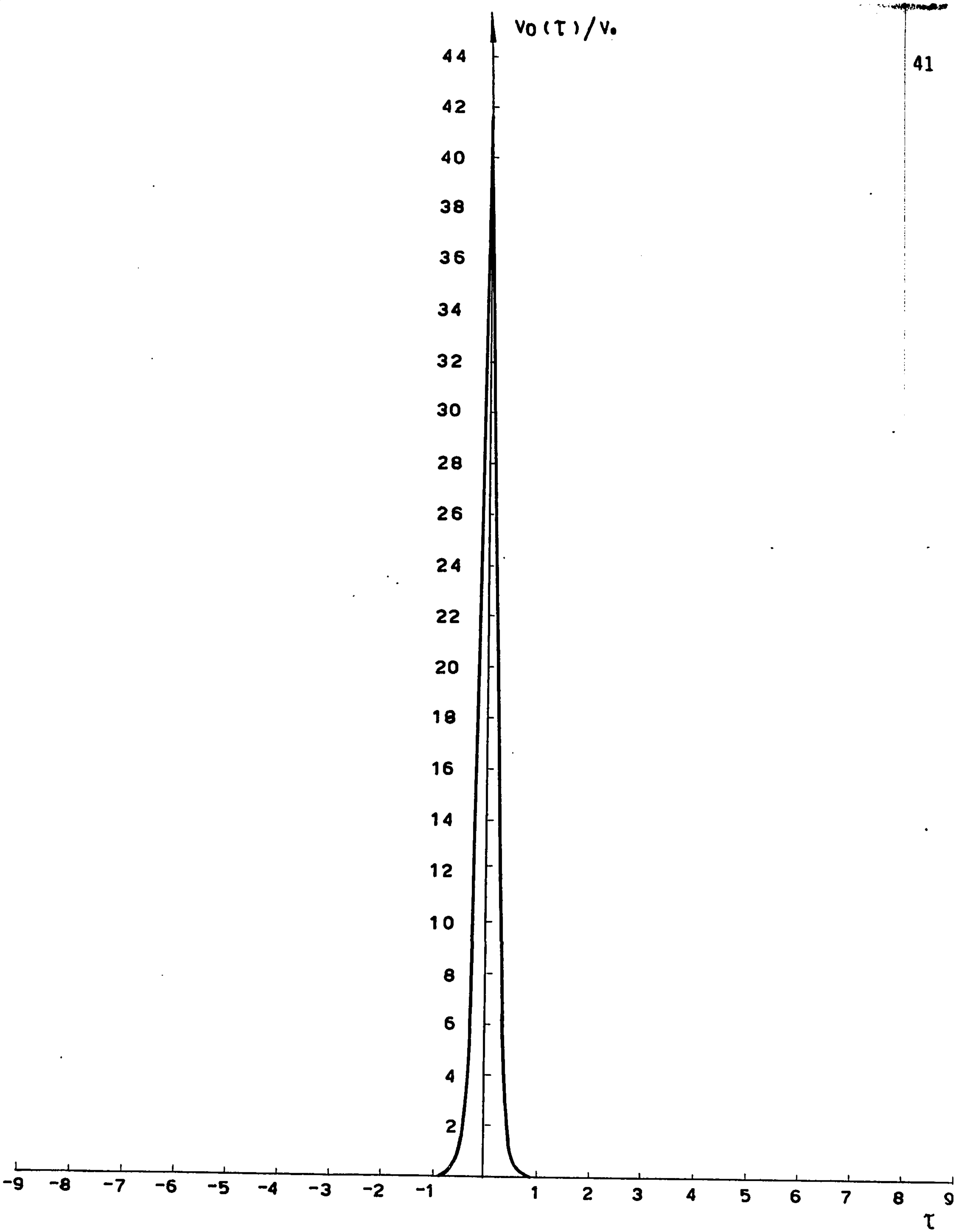

FIG. 3.10 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = 0$ ($\Omega_1^2 = \Omega_2^2 = 13{\cdot}15$)

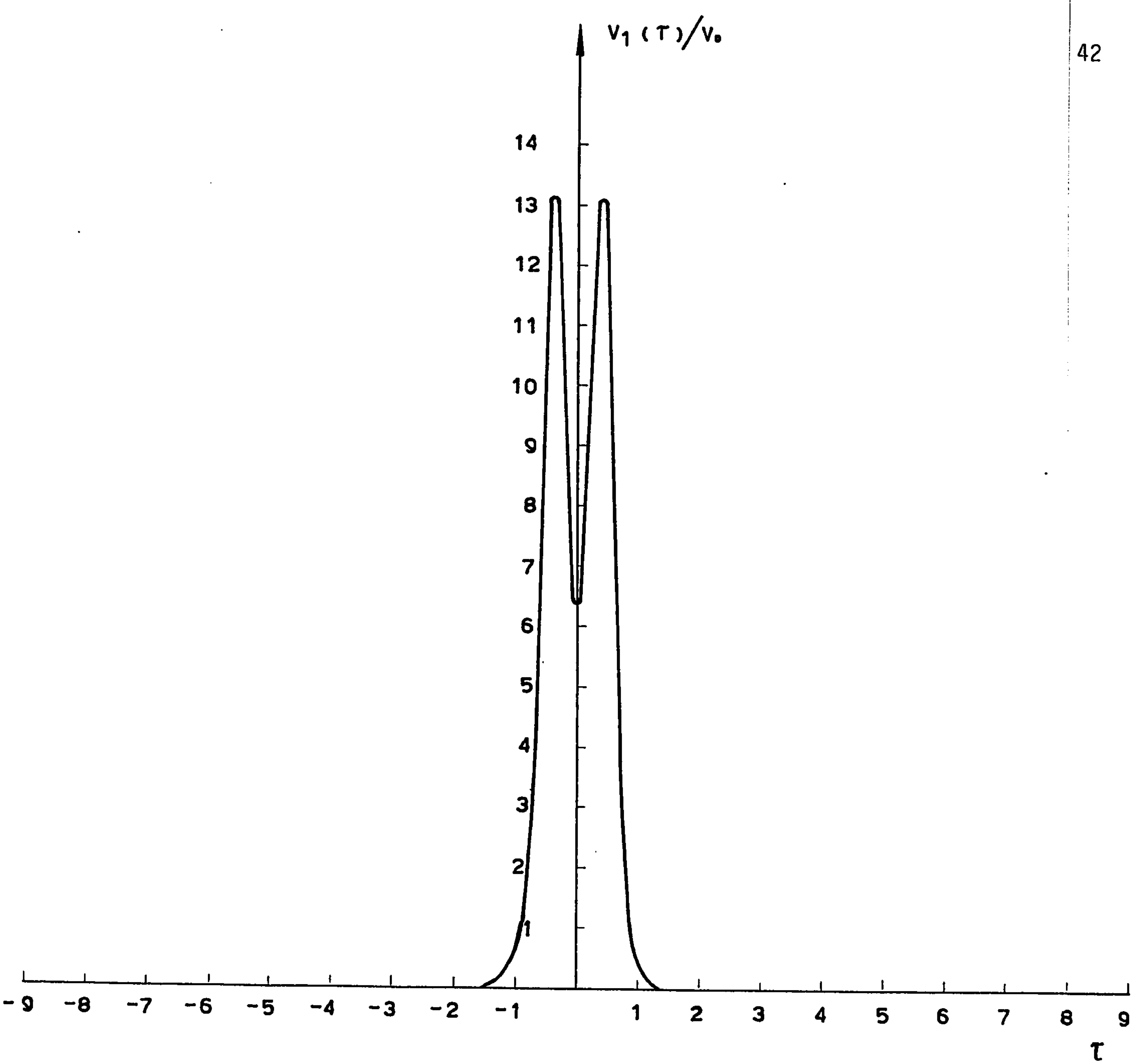

FIG. 3.11 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $h = 1$ ($\Omega_1^2 = \Omega_2^2 = 13.15$)

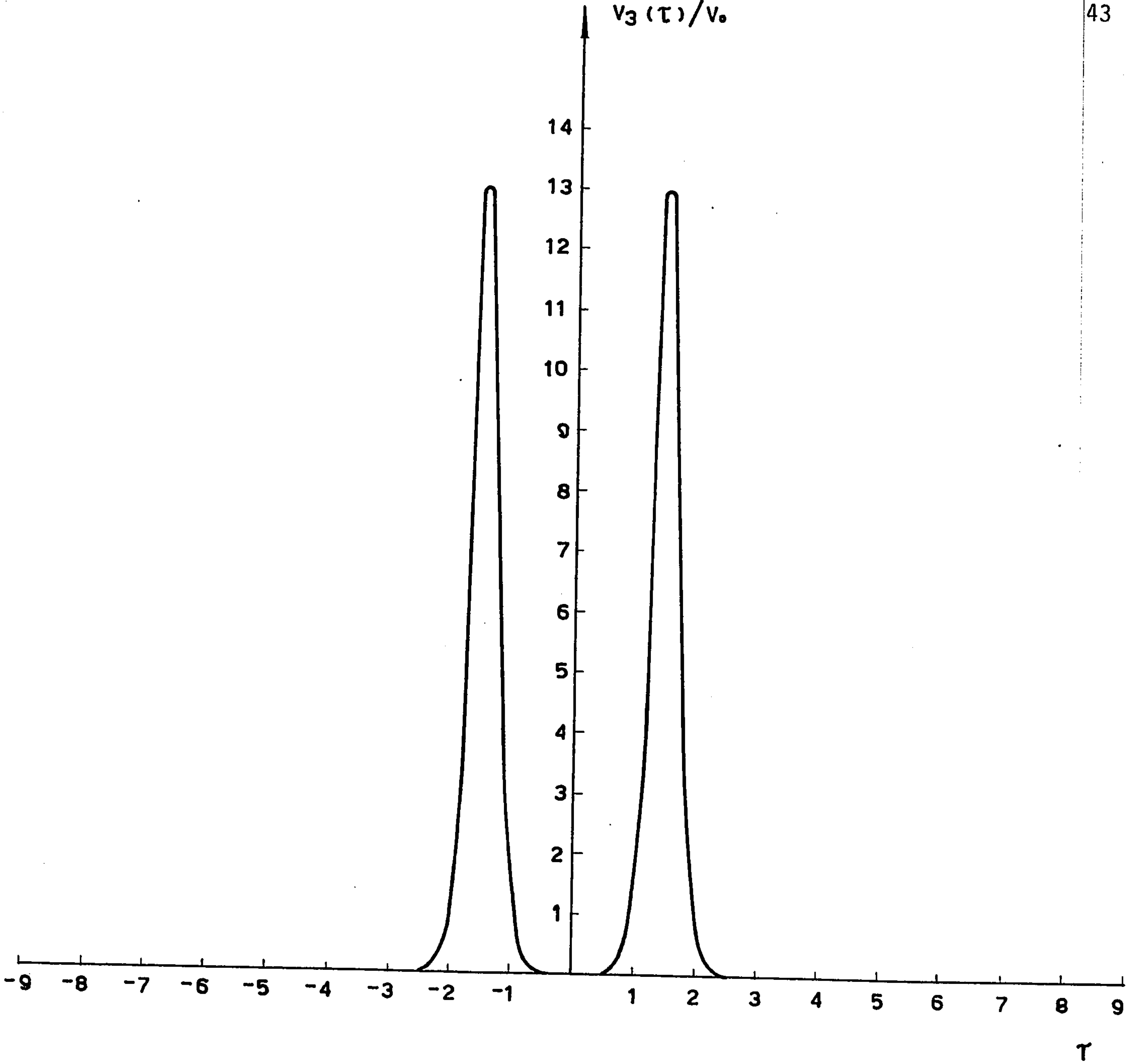

FIG. 3·12 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = 3$ ($\Omega_1^2 = \Omega_2^2 = 13.15$)

3.5 Collision of Two Voltage Solitons Moving in the same Direction

The motion of two voltage solitons moving in the same direction is expressed by Eq. (2.26), (2.27), (2.28), (2.31), (2.32), and (2.33) with $\Omega_1^2 = 13.15$ and $\Omega_2^2 = 5.64$.

A computer program was written (see Appendix E) to carry on the computations. The results of which are tabulated in Table (3.6) for n=-10, -5,0, and 5. These results are plotted in Fig. (3.13) to (3.16).

The plots show that the larger soliton is accelerated when it approaches the smaller, and at the same time the latter is correspondingly retarded by the former. The larger one absorbs the smaller with a decrease in amplitude and finally releases it. Also, they emerge after collision without a change in shape.

τ	$V_{-10}(\tau)/V_0$	τ	$V_{-5}(\tau)/V_0$	τ	$V_0(\tau)/V_0$	τ	$V\ (\tau)/V_0$
-8.6	0.0469	-5.2	0.0546	-1.9	0.0403	1.4	0.0692
-8.5	0.0752	-5.1	0.0875	-1.8	0.0651	1.5	0.1294
-8.4	0.1205	-5.0	0.1400	-1.7	0.1050	1.6	0.2449
-8.3	0.1925	-4.9	0.2235	-1.6	0.1695	1.7	0.4672
-8.2	0.3063	-4.8	0.3551	-1.5	0.2732	1.8	0.8932
-8.1	0.4843	-4.7	0.5602	-1.4	0.4393	1.9	1.6942
-8.0	0.7584	-4.6	0.8738	-1.3	0.7029	2.0	3.1316
-7.9	1.1694	-4.5	1.3391	-1.2	1.1142	2.1	5.4725
-7.8	1.7610	-4.4	1.9975	-1.1	1.7372	2.2	8.6209
-7.7	2.5592	-4.3	2.8622	-1.0	2.6352	2.3	11.5108
-7.6	3.5325	-4.2	3.8730	-0.9	3.8274	2.4	12.3369
-7.5	4.5432	-4.1	4.8518	-0.8	5.2116	2.5	10.4639
-7.4	5.3374	-4.0	5.5210	-0.7	6.4986	2.6	7.2954
-7.3	5.6403	-3.9	5.6349	-0.6	7.2768	2.7	4.4728
-7.2	5.3302	-3.8	5.1493	-0.5	7.2682	2.8	2.5907
-7.1	4.5316	-3.7	4.2568	-0.4	6.5623	2.9	1.5286
-7.0	3.5202	-3.6	3.2458	-0.3	5.5425	3.0	1.0121
-6.9	2.5487	-3.5	2.3394	-0.2	4.6050	3.1	0.8384
-6.8	1.7534	-3.4	1.6410	-0.1	3.9857	3.2	0.9017
-6.7	1.1649	-3.3	1.1671	0.0	3.7735	3.3	1.1673
-6.6	0.7568	-3.2	0.9016	0.1	3.9860	3.4	1.6413
-6.5	0.4860	-3.1	0.8384	0.2	4.6055	3.5	2.3398
-6.4	0.3124	-3.0	1.0123	0.3	5.5432	3.6	3.2463
-6.3	0.2061	-2.9	1.5290	0.4	6.5630	3.7	4.2572
-6.2	0.1481	-2.8	2.5915	0.5	7.2685	3.8	5.1496
-6.1	0.1307	-2.7	4.4740	0.6	7.2765	3.9	5.6350
-6.0	0.1584	-2.6	7.2971	0.7	6.4979	4.0	5.5208
-5.9	0.2544	-2.5	10.4655	0.8	5.2107	4.1	4.8514
-5.8	0.4737	-2.4	12.3372	0.9	3.8265	4.2	3.8726
-5.7	0.9278	-2.3	11.5096	1.0	2.6345	4.3	2.8619
-5.6	1.8211	-2.2	8.6191	1.1	1.7367	4.4	1.9972
-5.5	3.4690	-2.1	5.4709	1.2	1.1138	4.5	1.3389
-5.4	6.1730	-2.0	3.1305	1.3	0.7027	4.6	0.8737
-5.3	9.7079	-1.9	1.6936	1.4	0.4392	4.7	0.5601
-5.2	12.5952	-1.8	0.8929	1.5	0.2731	4.8	0.3551
-5.1	12.7690	-1.7	0.4670	1.6	0.1694	4.9	0.2235
-5.0	10.0822	-1.6	0.2448	1.7	0.1050	5.0	0.1400
-4.9	6.5187	-1.5	0.1294	1.8	0.0651	5.1	0.0875
-4.8	3.6962	-1.4	0.0692	1.9	0.0403	5.2	0.0546
-4.7	1.9439						
-4.6	0.9827						
-4.5	0.4871						
-4.4	0.2393						
-4.3	0.1171						
-4.2	0.0573						

Table 3.6 Theoretical results expressing two voltage solitons moving in the same direction for n=-10,-5,0,5; (Ω_1^2 = 13.15, Ω_2^2 = 5.64)

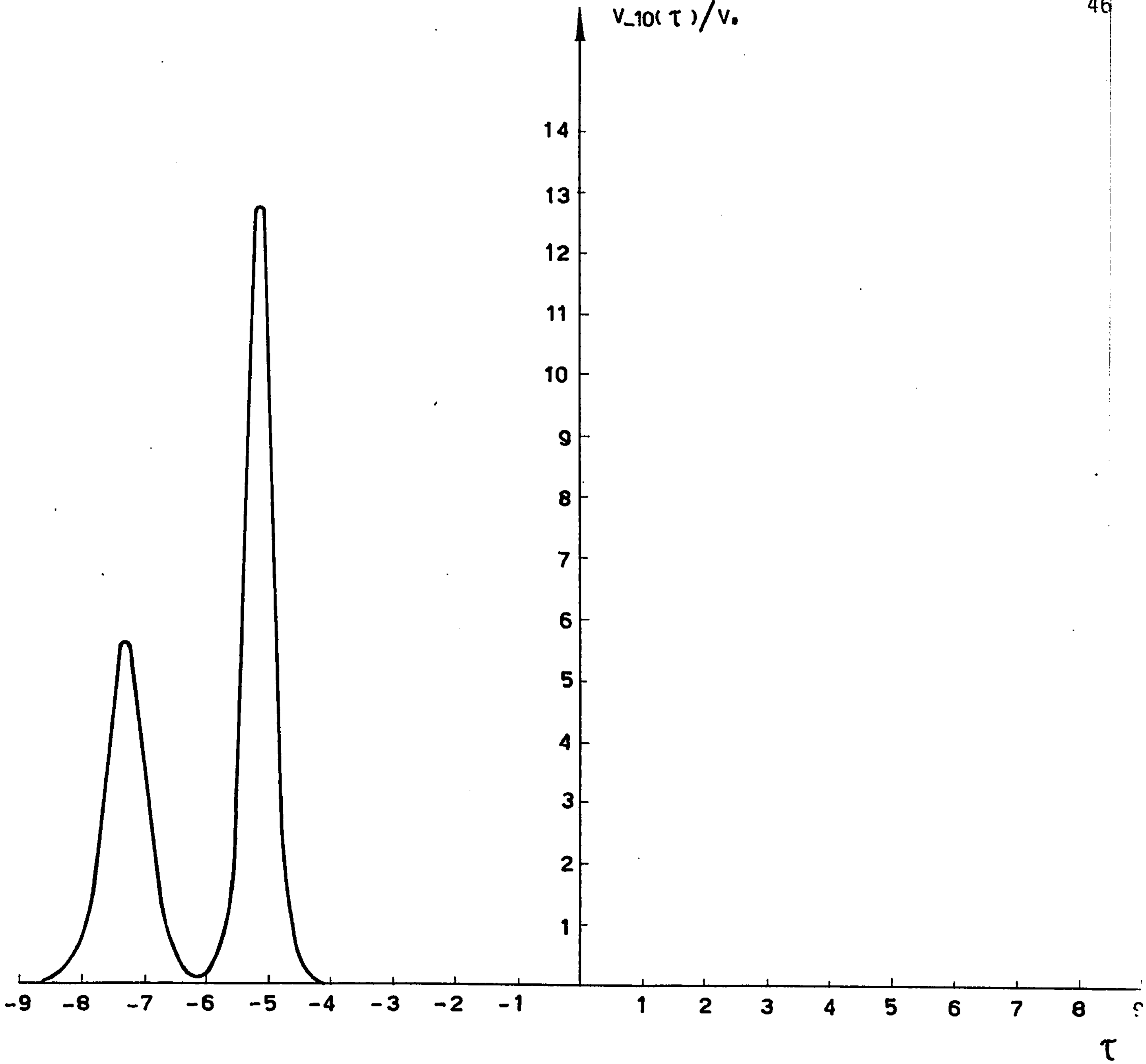

FIG. 3.13 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN THE SAME DIRECTION FOR $n = -10$ ($\Omega_1^2 = 13.15$, $\Omega_2^2 = 5{\cdot}64$)

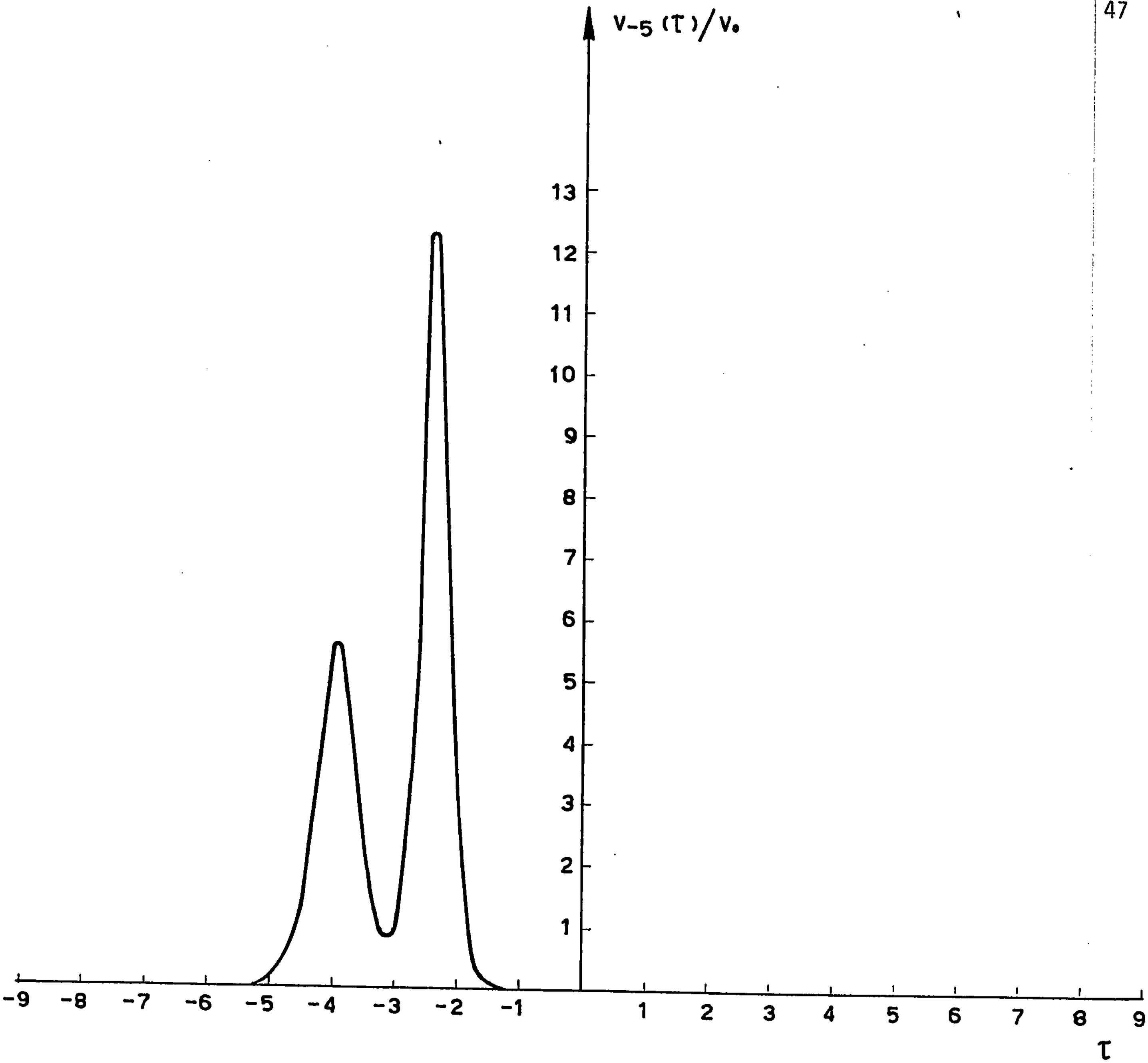

FIG. 3.14 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN THE SAME DIRECTION FOR $n = -5$ ($\Omega_1^2 = 13.15$, $\Omega_2^2 = 5.64$)

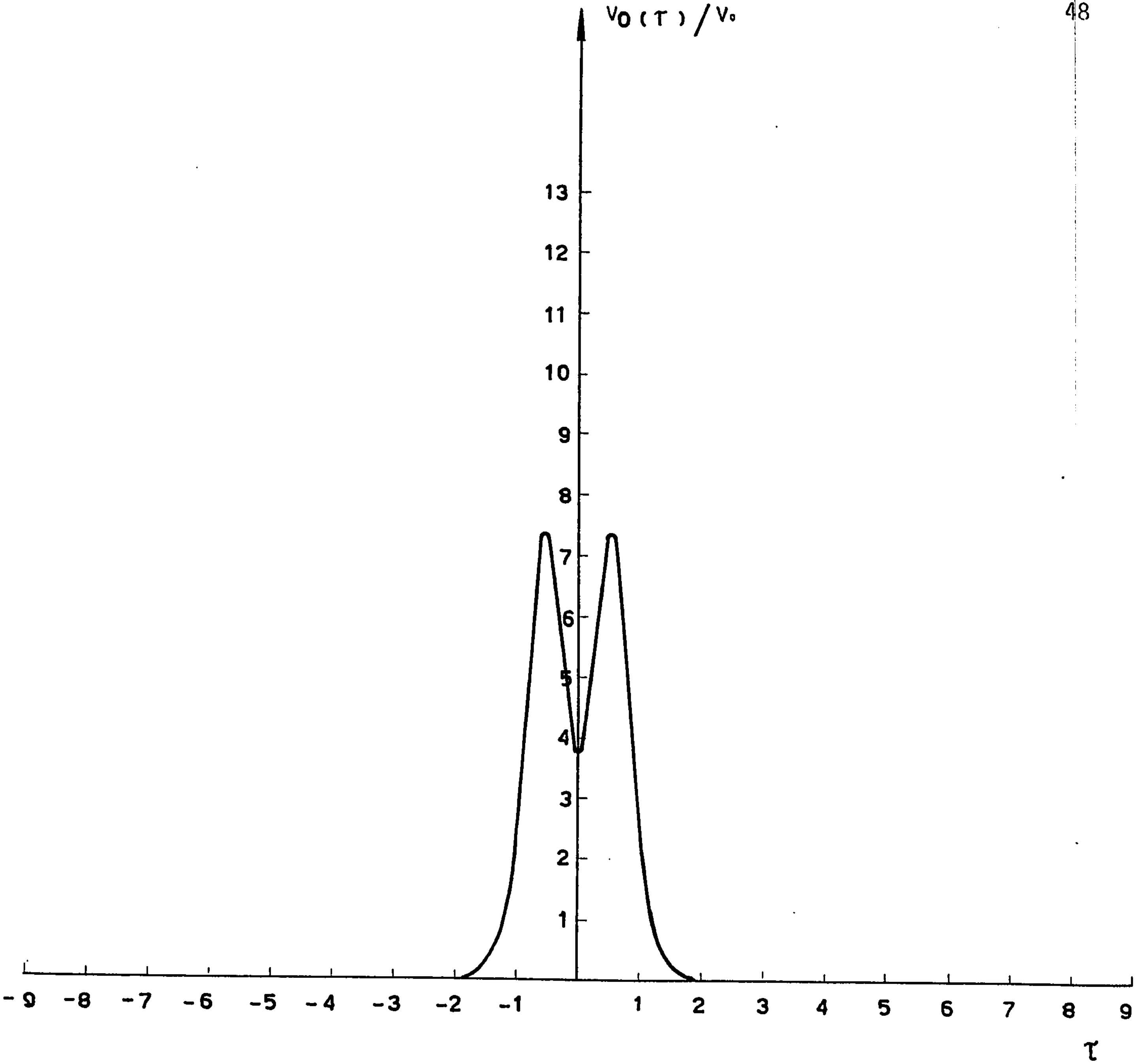

FIG. 3.15 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN THE SAME DIRECTION FOR $h = 0$ ($\Omega_1^2 = 13.15$, $\Omega_2^2 = 5.64$)

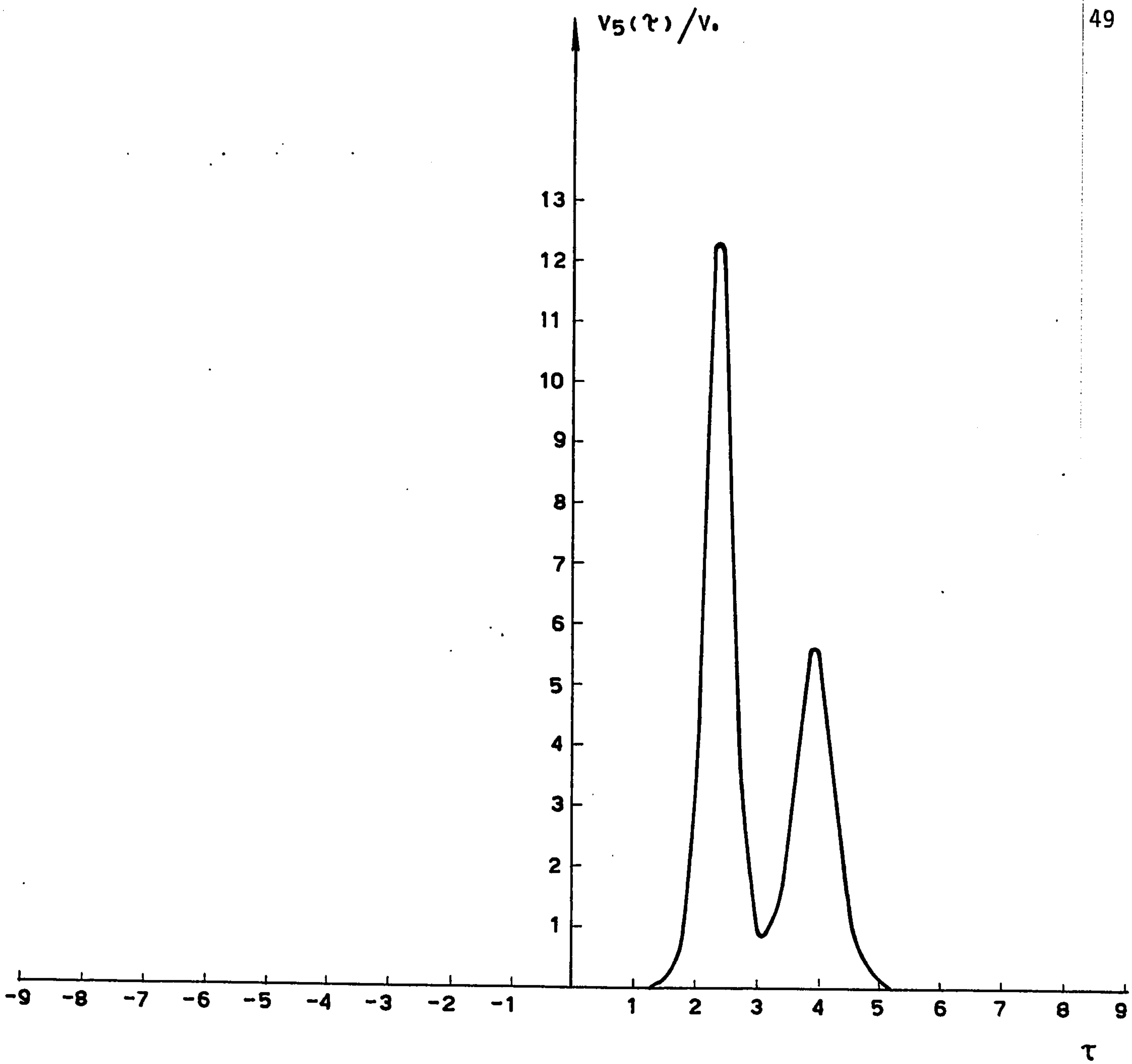

FIG. 3.16 TWO NORMALIZED VOLTAGE SOLITONS MOVING IN THE SAME DIRECTION FOR $n = 5$ ($\Omega_1^2=13.15$, $\Omega_2^2= 5.64$)

3.6 Conservation of The Voltage Integral

In this section we shall investigate the assumption made by Hirota (24) which say that,

$$\int_{-\infty}^{\infty} \frac{V_n(\tau)}{V_o} \, d\tau \quad = \quad \text{constant, independent of position n} \qquad (3.6)$$

A computer program was written (see Appendix F) to compute the above integral for n=-10 to 7 for two voltage solitons moving in the same direction. The results are tabulated in Table (3.7).

From these results it is found that the voltage integral is constant and has the value of 12.002 independent of position n.

The same was applied to two voltage solitons moving in opposite directions. Where a computer program was written (see Appendix G) to compute the voltage integral for n=-10 to 10. The results are tabulated in Table (3.8). We notice again that the voltage integral is constant of value of 14.505 independent of position n.

n	$\int (V_n(\tau)/V_o)\, d\tau$
-10	12.002
-9	12.002
-8	12.002
-7	12.002
-6	12.002
-5	12.002
-4	12.002
-3	12.002
-2	12.002
-1	12.002
0	12.002
1	12.002
2	12.002
3	12.002
4	12.002
5	12.002
6	12.002
7	12.002

Table 3.7 Integral of $V_n\,(\tau)/V_o$ with respect to τ of two voltage solitons moving in the same direction for n=-10 to n=7 (Ω_1^2 = 13.15, Ω_2^2 = 5.64)

n	$(V_n(\tau)/V_0)\ d\tau$
-10	14.505
-9	14.505
-8	14.505
-7	14.505
-6	14.505
-5	14.505
-4	14.505
-3	14.505
-2	14.505
-1	14.505
0	14.505
1	14.505
2	14.505
3	14.505
4	14.505
5	14.505
6	14.505
7	14.505
8	14.505
9	14.505
10	14.505

Table 3.8 Integral of $V_n\ (\tau)/V_0$ with respect to τ of two voltage solitons moving in opposite directions for n=-10 to n=10 ($\Omega_1^2 = \Omega_2^2 = 13.15$)

3.7 Retaining of Shape of a Single Current Soliton

From Eq. (2.1)

$$\frac{\partial}{\partial t} L \; I_n(\tau) = V_n(\tau) - V_{n+1}(\tau) \tag{3.7}$$

$$\frac{L}{V_o} \frac{\partial}{\partial t} \; I_n(\tau) = \frac{V_n(\tau)}{V_o} - \frac{V_{n+1}(\tau)}{V_o} \tag{3.8}$$

$$I_n(\tau) = \frac{V_o}{L} \int \left(\frac{V_n(\tau)}{V_o} - \frac{V_{n+1}(\tau)}{V_o} \right) d\tau \tag{3.9}$$

where

$$\frac{V_n(\tau)}{V_o} = \Omega^2 \; \mathrm{sech}^2 \; (\Omega\tau - Pn) \tag{3.10}$$

A computer program was written (see Appendix H) to compute the normalized current soliton as a function of τ for n=-10 to n=10. The results for n=-10,-6,6, and 10 are tabulated in Table (3.9). These results are plotted as shown in Fig. (3.17).

From the graphs we observe that the current soliton travels from left to right without changing shape.

τ	$I_{-10}(\tau)$	τ	$I_{-6}(\tau)$	τ	$I_{6}(\tau)$	τ	$I_{10}(\tau)$
-7.0	0.00014	-4.8	0.00013	1.8	0.00012	4.0	0.00011
-6.9	0.00029	-4.7	0.00028	1.9	0.00024	4.1	0.00023
-6.8	0.00059	-4.6	0.00057	2.0	0.00050	4.2	0.00048
-6.7	0.00123	-4.5	0.00117	2.1	0.00103	4.3	0.00099
-6.6	0.00253	-4.4	0.00243	2.2	0.00213	4.4	0.00204
-6.5	0.00523	-4.3	0.00501	2.3	0.00440	4.5	0.00421
-6.4	0.01078	-4.2	0.01033	2.4	0.00908	4.6	0.00869
-6.3	0.02223	-4.1	0.02131	2.5	0.01873	4.7	0.01792
-6.2	0.04577	-4.0	0.04386	2.6	0.03856	4.8	0.03690
-6.1	0.09386	-3.9	0.08999	2.7	0.07917	4.9	0.07578
-6.0	0.19112	-3.8	0.18333	2.8	0.16155	5.0	0.15471
-5.9	0.38352	-3.7	0.36832	2.9	0.32560	5.1	0.31213
-5.8	0.74820	-3.6	0.72013	3.0	0.64053	5.2	0.61525
-5.7	1.38610	-3.5	1.33931	3.1	1.20454	5.3	1.16111
-5.6	2.35781	-3.4	2.29208	3.2	2.09804	5.4	2.03402
-5.5	3.55619	-3.3	3.48427	3.3	3.26468	5.5	3.18980
-5.4	4.66933	-3.2	4.61260	3.4	4.43162	5.6	4.36720
-5.3	5.37075	-3.1	5.34460	3.5	5.25300	5.7	5.21751
-5.2	5.49660	-3.0	5.50604	3.6	5.52228	5.8	5.52314
-5.1	5.02429	-2.9	5.06772	3.7	5.18924	5.9	5.22631
-5.0	4.05033	-2.8	4.11766	3.8	4.31754	6.0	4.38294
-4.9	2.84254	-2.7	2.91365	3.9	3.13314	6.1	3.20794
-4.8	1.75016	-2.6	1.80660	4.0	1.98625	6.2	2.04942
-4.7	0.97439	-2.5	1.01075	4.1	1.12901	6.3	1.17149
-4.6	0.50851	-2.4	0.52903	4.2	0.59665	6.4	0.62125
-4.5	0.25582	-2.3	0.26656	4.3	0.30223	6.5	0.31530
-4.4	0.12623	-2.2	0.13164	4.4	0.1966[illegible]	6.6	0.15630
-4.3	0.06167	-2.1	0.06434	4.5	0.07325	6.7	0.07654
-4.2	0.02997	-2.0	0.03127	4.6	0.03563	6.8	0.03725
-4.1	0.01452	-1.9	0.01515	4.7	0.01727	6.9	0.01806
-4.0	0.00701	-1.8	0.00732	4.8	0.00834	7.0	0.00873

Table (3.9) Theoretical results expressing one current soliton for n=-10,-6,6, and 10.

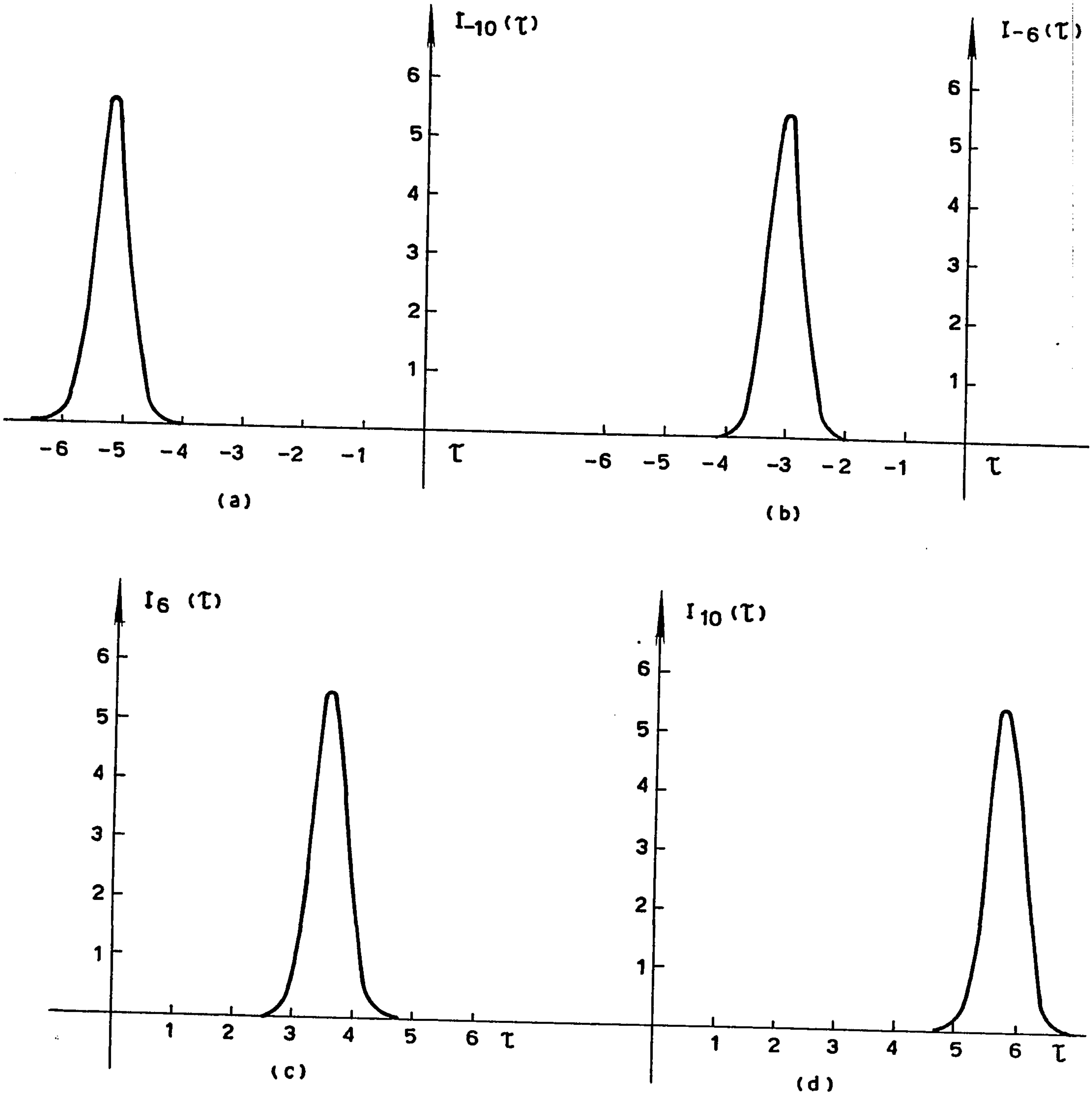

FIG. 3·17 NORMALIZED CURRENT SOLITON ; (a) n = -10 , (b) n = -6 , (c) n = 6, (d) n = 10 (Ω^2 = 13·15)

3.8 Collision of Two Current Solitons Moving in Opposite Directions

The motion of two current solitons moving in opposite directions is expressed by Eq. (2.26), (2.27), (2.28), (2.34), (2.35), (2.36), and (3.9), with $\Omega_1^2 = \Omega_2^2 = 13.15$.

A computer program was written (see Appendix I) to compute the magnitude of $I_n(\tau)$. The results of these computations are tabulated in Table (3.10) for n=-3, -1, 0,1, and 3. These results are plotted in Fig. (3.18) to (3.22).

The plots show that two current solitons collide and emerge after collision without changing their shapes. We also notice that the first current soliton has a positive magnitude and the second has a negative magnitude.

τ	$I_{-3}(\tau)$	τ	$I_{-1}(\tau)$	τ	$I_0(\tau)$	τ	$I_1(\tau)$	τ	$I_3(\tau)$
-2.1	0.06896	-1.0	0.06627	-1.0	-0.06627	-1.5	-0.09764	-2.6	-0.09972
-2.0	0.14094	-0.9	0.13544	-0.9	-0.13544	-1.4	-0.19870	-2.5	-0.20287
-1.9	0.28496	-0.8	0.27384	-0.8	-0.27384	-1.3	-0.39829	-2.4	-0.40640
-1.8	0.56396	-0.7	0.54193	-0.7	-0.54193	-1.2	-0.77535	-2.3	-0.79020
-1.7	1.07211	-0.6	1.02981	-0.6	-1.02981	-1.1	-1.43102	-2.2	-1.45544
-1.6	1.90054	-0.5	1.82173	-0.5	-1.82173	-1.0	-2.42014	-2.1	-2.45374
-1.5	3.02969	-0.4	2.88017	-0.4	-2.88017	-0.9	-3.62324	-2.0	-3.65899
-1.4	4.22479	-0.3	3.90112	-0.3	-3.90112	-0.8	-4.72097	-1.9	-4.74822
-1.3	5.13421	-0.2	4.29578	-0.2	-4.29578	-0.7	-5.39288	-1.8	-5.40476
-1.2	5.51684	-0.1	3.25630	-0.1	-3.25630	-0.6	-5.48408	-1.7	-5.47936
-1.1	5.29792	0.0	0.34443	0.0	-0.34443	-0.5	-4.97615	-1.6	-4.95875
-1.0	4.51760	0.1	-2.81961	0.1	2.81961	-0.4	-3.96938	-1.5	-3.95216
-0.9	3.36727	0.2	-4.23265	0.2	4.23265	-0.3	-2.73247	-1.4	-2.74121
-0.8	2.18743	0.3	-4.05077	0.3	4.05077	-0.2	-1.59833	-1.3	-1.67113
-0.7	1.26593	0.4	-3.09847	0.4	3.09847	-0.1	-0.72960	-1.2	-0.92407
-0.6	0.67642	0.5	-2.01364	0.5	2.01364	0.0	-0.06116	-1.1	-0.48030
-0.5	0.34457	0.6	-1.15973	0.6	1.15973	0.1	0.58766	-1.0	-0.24110
-0.4	0.17079	0.7	-0.61720	0.7	0.61720	0.2	1.40643	-0.9	-0.11883
-0.3	0.08296	0.8	-0.31380	0.8	0.31380	0.3	2.49327	-0.8	-0.05802
-0.2	0.03885	0.9	-0.15570	0.9	0.15570	0.4	3.73390	-0.7	-0.02818
-0.1	0.01570	1.0	-0.07631	1.0	0.07631	0.5	4.81214	-0.6	-0.01365
0.0	0.00115	1.1	-0.03717	1.1	0.03717	0.6	5.43069	-0.5	-0.00658
0.1	-0.01271	1.2	-0.01806	1.2	0.01806	0.7	5.45880	-0.4	-0.00315
0.2	-0.03341	1.3	-0.00876	1.3	0.00876	0.8	4.89406	-0.3	-0.00149
0.3	-0.07205	1.4	-0.00425	1.4	0.00425	0.9	3.85891	-0.2	-0.00066
0.4	-0.14873	1.5	-0.00207	1.5	0.00207	1.0	2.64747	-0.1	-0.00024
0.5	-0.30101	1.6	-0.00101	1.6	0.00101	1.1	1.59948	0.0	0.00003
0.6	-0.59456	1.7	-0.00049	1.7	0.00049	1.2	0.87905	0.1	0.00029
0.7	-1.12544	1.8	-0.00024	1.8	0.00024	1.3	0.45526	0.2	0.00067
0.8	-1.98093	1.9	-0.00012	1.9	0.00012	1.4	0.22806	0.3	0.00139
0.9	-3.12679	2.0	-0.00007	2.0	0.00007	1.5	0.11226	0.4	0.00284
1.0	-4.31191	2.1	-0.00004	2.1	0.00004	1.6	0.05475	0.5	0.00581
1.1	-5.18591	2.2	-0.00002	2.2	0.00002	1.7	0.02655	0.6	0.01194
1.2	-5.52193	2.3	-0.00002	2.3	0.00002	1.8	0.01281	0.7	0.02456
1.3	-5.25567	2.4	-0.00001	2.4	0.00001	1.9	0.00614	0.8	0.05048
1.4	-4.43671	2.5	-0.00001	2.5	0.00001	2.0	0.00291	0.9	0.10340
1.5	-3.27069	2.6	-0.00001	2.6	0.00001	2.1	0.00134	1.0	0.21017
1.6	-2.10318					2.2	0.00058	1.1	0.42051
1.7	-1.20798					2.3	0.00021	1.2	0.81590
1.8	-0.64245					2.4	0.00004	1.3	1.49736
1.9	-0.32652					2.5	-0.00005	1.4	2.51081
2.0	-0.16191					2.6	-0.00009	1.5	3.71894
2.1	-0.07924					2.7	-0.00011	1.6	4.79316
						2.8	-0.00012	1.7	5.42312
						2.9	-0.00013	1.8	5.46795
						3.0	-0.00013	1.9	4.91970
								2.0	3.89518
								2.1	2.68362
								2.2	1.62699
								2.3	0.89632
								2.4	0.46489
								2.5	0.23313
								2.6	0.11486
								2.7	0.05609

Table (3.10) Theoretical results expressing two current solitons moving in opposite directions for n=-3,-1,0,1, and 3.

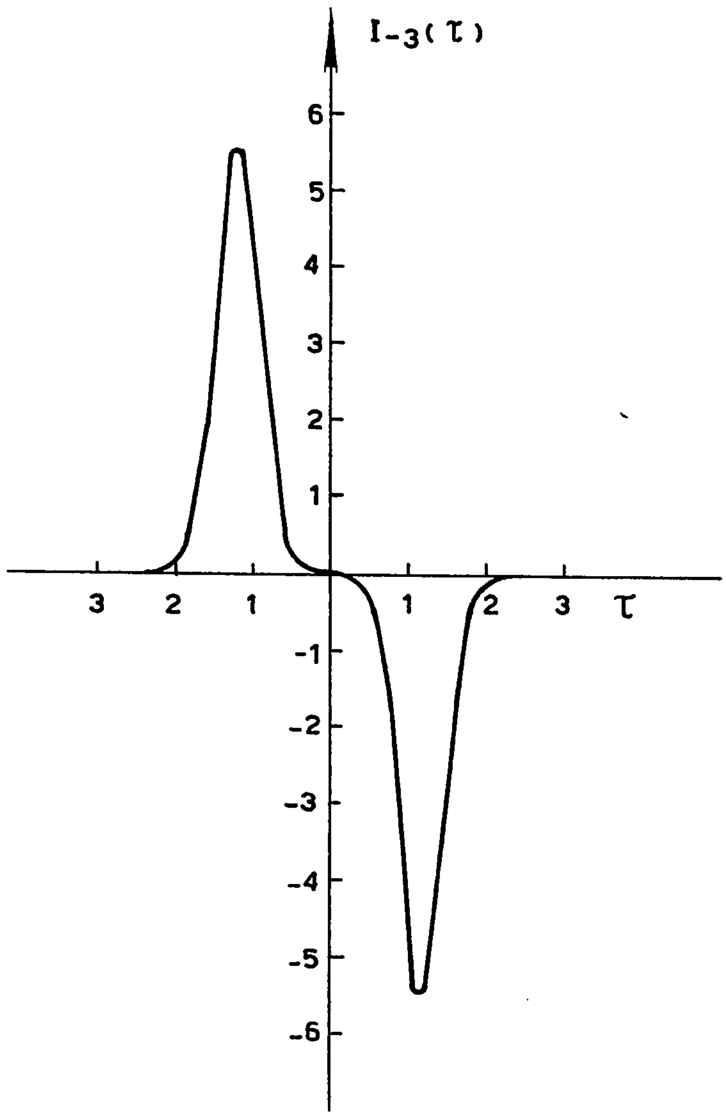

FIG. 3·18 NORMALIZED CURRENT SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = -3$ ($\Omega_1^2 = \Omega_2^2 = 13 \cdot 15$)

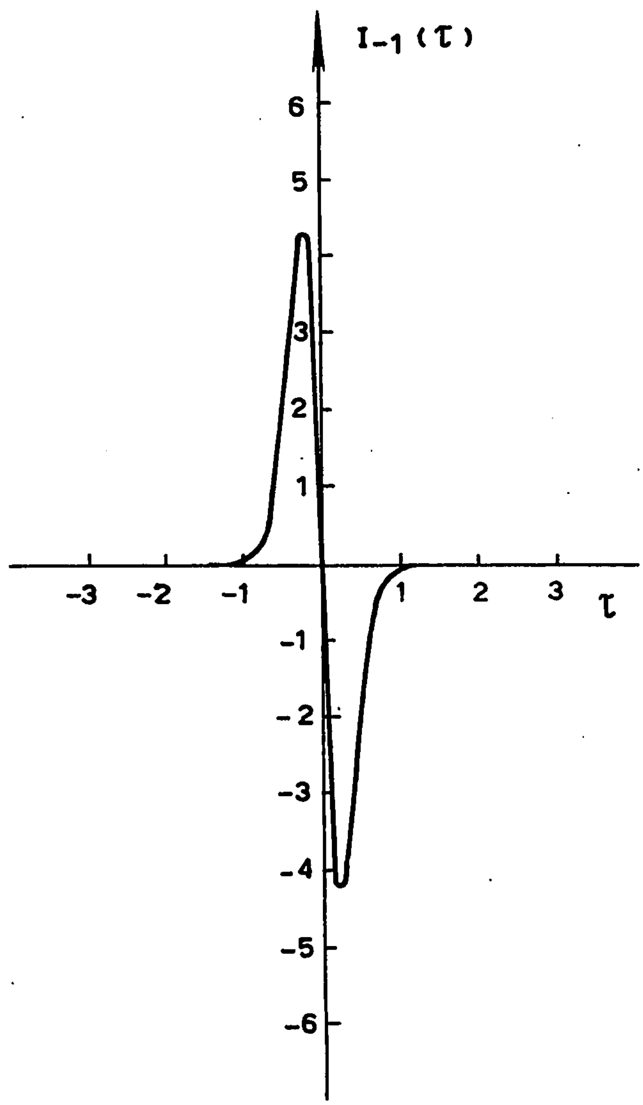

FIG. 3.19 NORMALIZED CURRENT SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n=-1$ ($\omega_1^2 = \omega_2^2 = 13.15$)

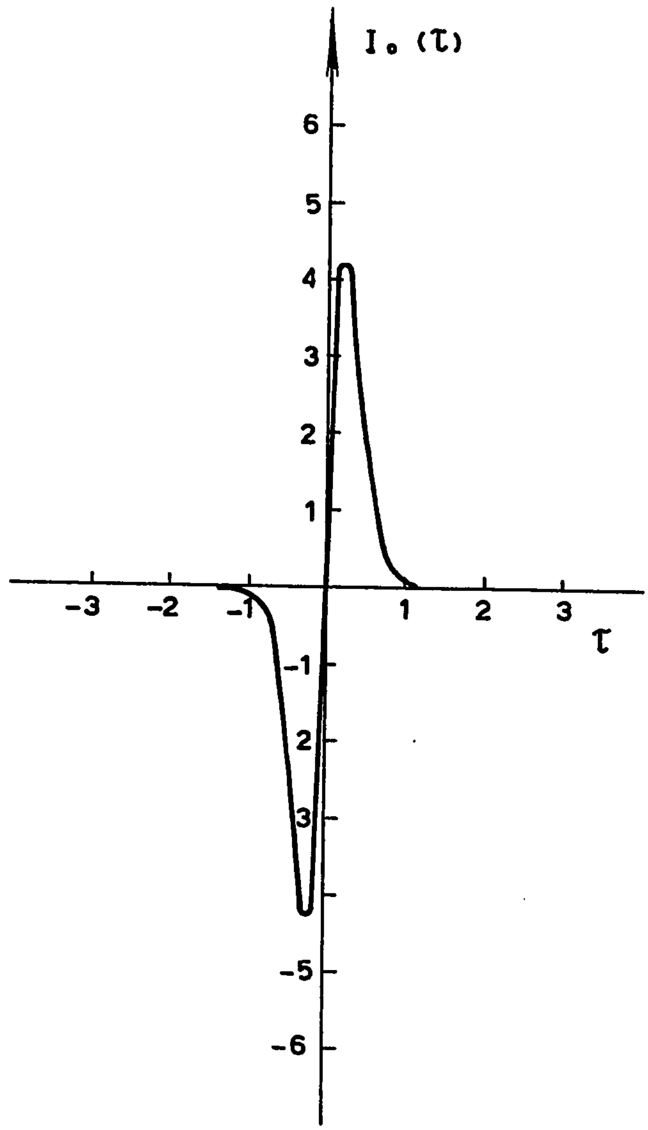

FIG. 3.20 NORMALIZED CURRENT SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = 0$ $(\Omega_1^2 = \Omega_2^2 = 13.15)$

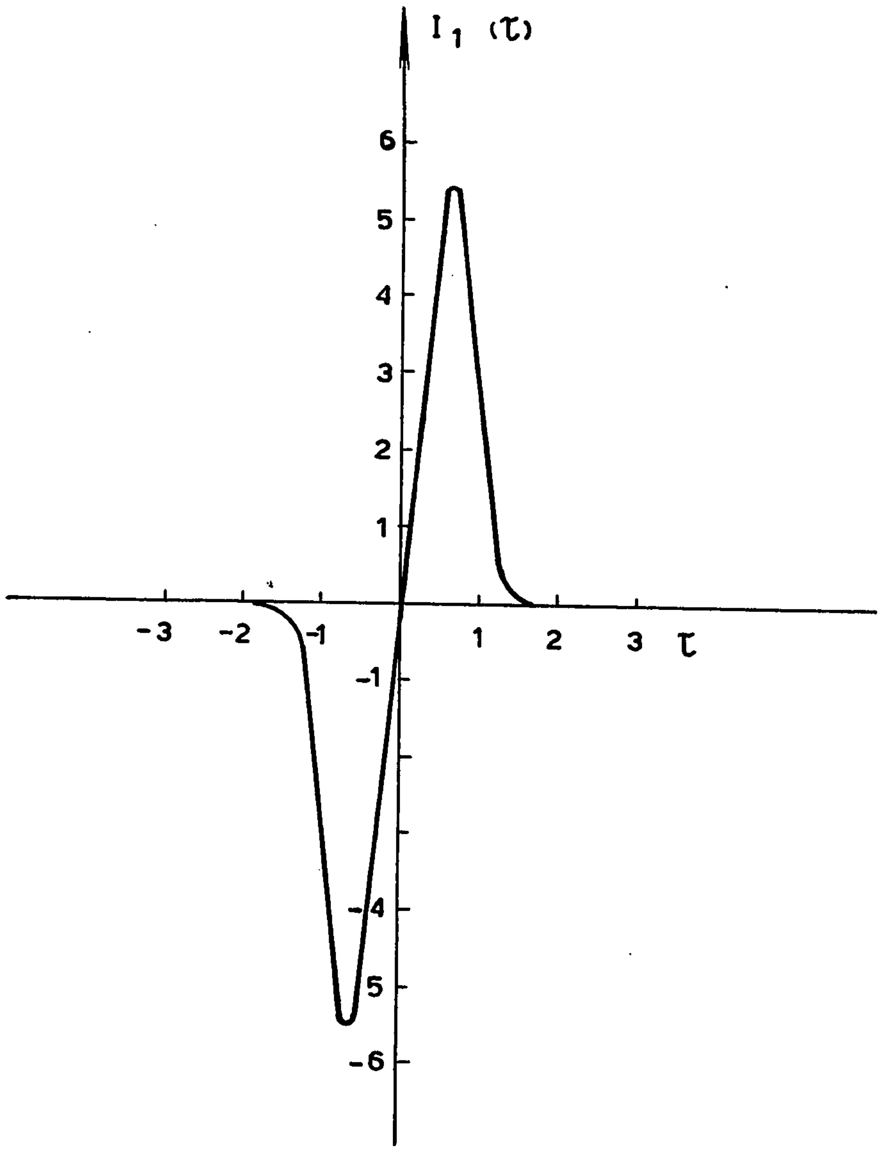

FIG. 3.21 NORMALIZED CURRENT SOLITONS MOVING IN OPPOSITE DIRECTIONS FOR $n = 1$ ($\sim_1^2 = \sim_2^2 = 13.15$)

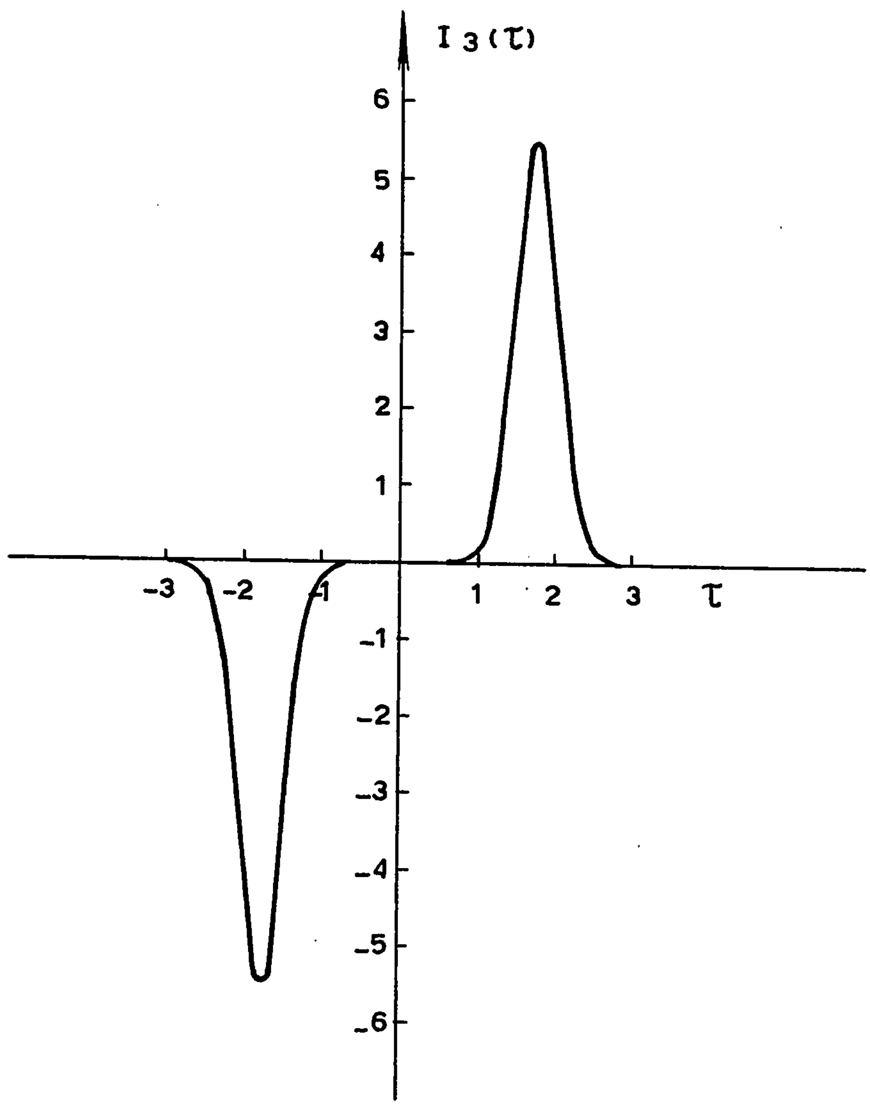

FIG. 3.22 NORMALIZED CURRENT SOLITONS MOVING IN OPPOSITE DIRECTION FOR $n = 3$ ($\sim_1^2 = \sim_2^2 = 13.15$)

3.9 Collision of Two Current Solitons Moving in The Same Direction

The motion of two current solitons moving in the same direction is expressed by Eq.(2.26),(2.27),(2.28),(2.31),(2.32),(2.33), and (3.9) with Ω_1^2 =13.15 and Ω_2^2 =5.64.

A computer program was written (see Appendix J) to compute the magnitude of I_n (τ).
The results of these computatinos are tabulated in Table (3.11) for n=-8,-5,0, and 8. These results are plotted in Fig.(3.23) to (3.26).

The plots show that the larger soliton is accelerated when it approaches the smaller, and at the same time the latter is correspondingly retarded by the former. The larger soliton absorbs the smaller with a decrease in amplitude and finally releases it. Also, they emerge after collision without a chang in shape.

τ	$I_{-8}(\tau)$	τ	$I_{-5}(\tau)$	τ	$I_0(\tau)$	τ	$I_8(\tau)$
-6.8	0.07658	-4.8	0.06953	-1.5	0.05234	3.4	0.06722
-6.7	0.12184	-4.7	0.11075	-1.4	0.08437	3.5	0.13517
-6.6	0.19265	-4.6	0.17541	-1.3	0.13575	3.6	0.26999
-6.5	0.30167	-4.5	0.27538	-1.2	0.21761	3.7	0.53011
-6.4	0.46545	-4.4	0.42651	-1.1	0.34634	3.8	1.00472
-6.3	0.70250	-4.3	0.64733	-1.0	0.54437	3.9	1.78647
-6.2	1.02748	-4.2	0.95414	-0.9	0.83826	4.0	2.87519
-6.1	1.44043	-4.1	1.35119	-0.8	1.25081	4.1	4.06677
-6.0	1.91461	-4.0	1.81813	-0.7	1.78487	4.2	5.01898
-5.9	2.39279	-3.9	2.30354	-0.6	2.40424	4.3	5.47310
-5.8	2.80028	-3.8	2.73436	-0.5	3.02864	4.4	5.33605
-5.7	3.06865	-3.7	3.03883	-0.4	3.55563	4.5	4.63183
-5.6	3.15407	-3.6	3.16712	-0.3	3.89591	4.6	3.52640
-5.5	3.04326	-3.5	3.09973	-0.2	3.99544	4.7	2.34468
-5.4	2.75369	-3.4	2.84829	-0.1	3.84295	4.8	1.39264
-5.3	2.33316	-3.3	2.45590	0.0	3.48057	4.9	0.77421
-5.2	1.85253	-3.2	1.99264	0.1	3.01420	5.0	0.43115
-5.1	1.38591	-3.1	1.53922	0.2	2.59265	5.1	0.26797
-5.0	0.98776	-3.0	1.16525	0.3	2.34894	5.2	0.21477
-4.9	0.68255	-2.9	0.91911	0.4	2.34695	5.3	0.23503
-4.8	0.47127	-2.8	0.83783	0.5	2.56645	5.4	0.31690
-4.7	0.34569	-2.7	0.96734	0.6	2.91855	5.5	0.46434
-4.6	0.30227	-2.6	1.37271	0.7	3.27967	5.6	0.68972
-4.5	0.35392	-2.5	2.10662	0.8	3.53513	5.7	1.00525
-4.4	0.54120	-2.4	3.11594	0.9	3.61056	5.8	1.41092
-4.3	0.93939	-2.3	4.16831	1.0	3.47794	5.9	1.88143
-4.2	1.63963	-2.2	4.95112	1.1	3.14917	6.0	2.36113
-4.1	2.66539	-2.1	5.25727	1.2	2.67159	6.1	2.77563
-4.0	3.85090	-2.0	5.02902	1.3	2.12112	6.2	3.05525
-3.9	4.86244	-1.9	4.31678	1.4	1.58173	6.3	3.15407
-3.8	5.41270	-1.8	3.29227	1.5	1.11738	6.4	3.05659
-3.7	5.38309	-1.7	2.23188	1.6	0.75643	6.5	2.77803
-3.6	4.78056	-1.6	1.37171	1.7	0.49647	6.6	2.36403
-3.5	3.73945	-1.5	0.78872	1.8	0.31902	6.7	1.88407
-3.4	2.55498	-1.4	0.43718	1.9	0.20217	6.8	1.41240
-3.3	1.54653	-1.3	0.23874	2.0	0.12698	6.9	1.00451
-3.2	0.85696	-1.2	0.13021	2.1	0.07929	7.0	0.68523
-3.1	0.44999	-1.1	0.07148	2.2	0.04931	7.1	0.45326
-3.0	0.22974	-1.0	0.03965	2.3	0.03055	7.2	0.29342
-2.9	0.11587	-0.9	0.02226	2.4	0.01886	7.3	0.18721
-2.8	0.05826	-0.8	0.01265	2.5	0.01158	7.4	0.11830
-2.7	0.02935	-0.7	0.00726	2.6	0.00705	7.5	0.07429

Table (3.11) Theoretical results expressing two current solitons moving in the same direction for n=-8,-5,0, and 8.

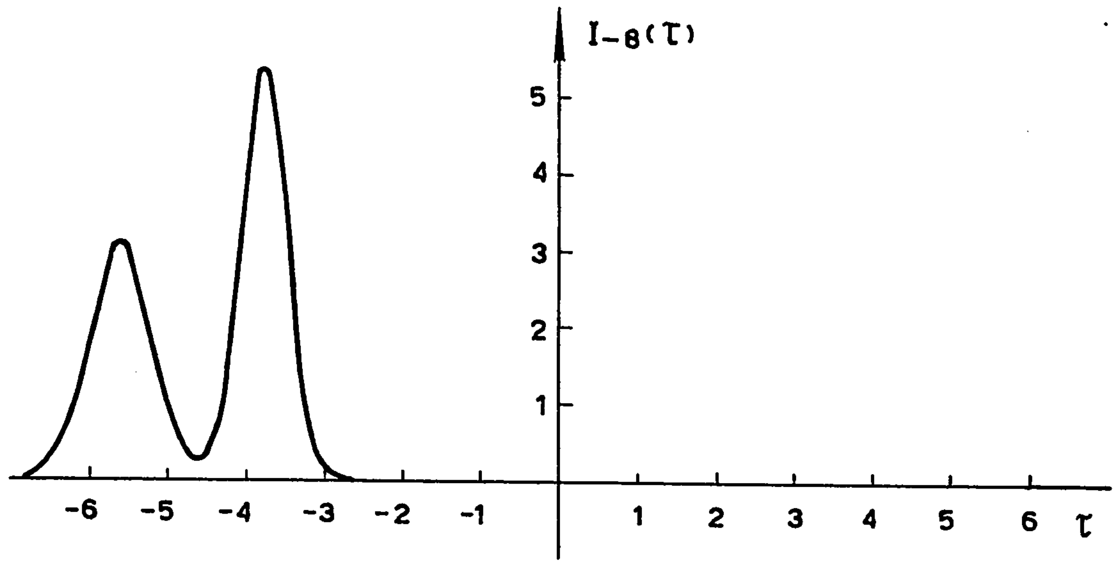

FIG. 3.23 NORMALIZED CURRENT SOLITONS MOVING IN THE SAME DIRECTION FOR n =-8 ($\omega_1^2 = 13.15$, $\omega_2^2 = 5.64$)

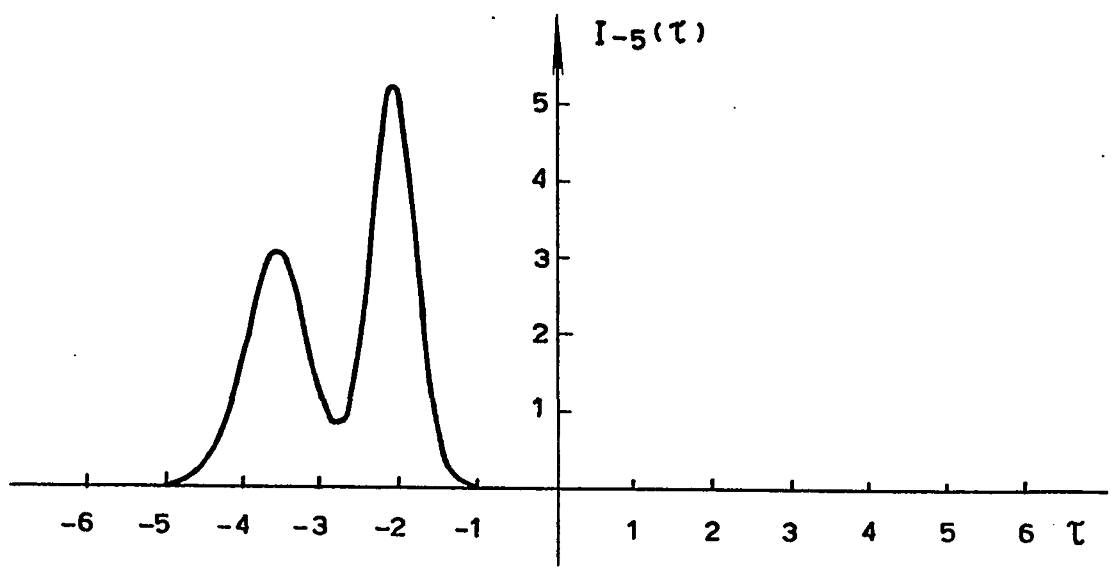

FIG.3.24 NORMALIZED CURRENT SOLITONS MOVING IN THE SAME DIRECTION FOR n= -5 ($\omega_1^2 = 13.15$, $\omega_2^2 = 5.64$)

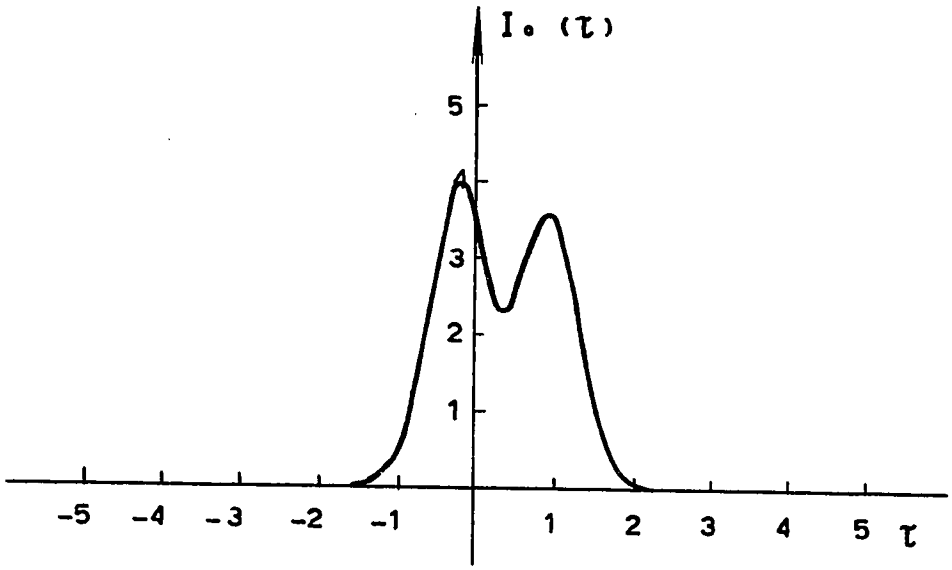

FIG.3.25 NORMALIZED CURRENT SOLITONS MOVING IN THE SAME DIRECTION FOR $n = 0$ ($\Omega_1^2 = 13.15$, $\Omega_2^2 = 5.64$)

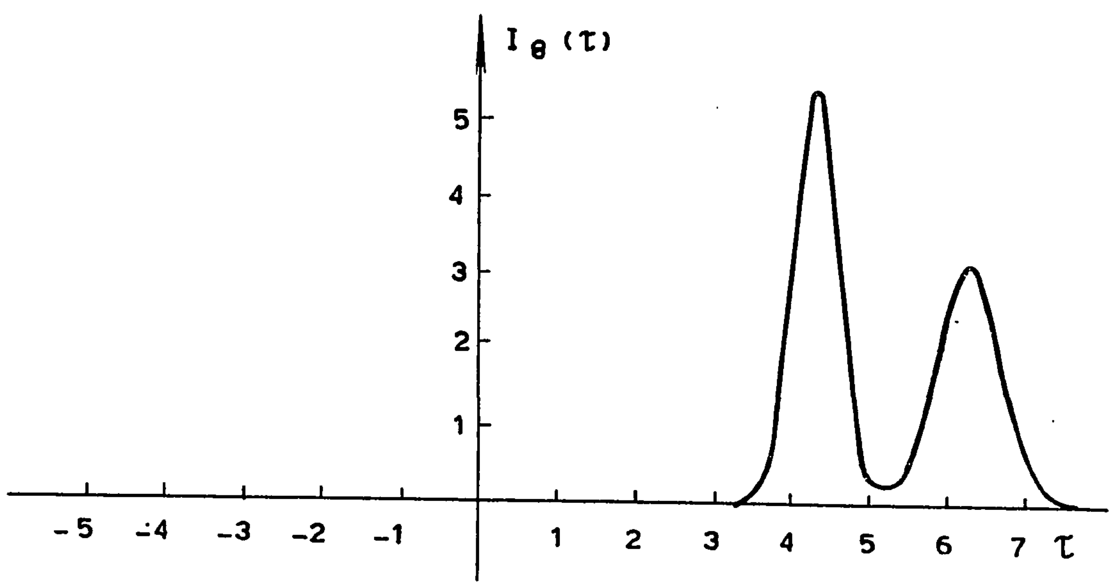

FIG. 3.26 NORMALIZED CURRENT SOLITONS MOVING IN THE SAME DIRECTION FOR $n = 8$ ($\Omega_1^2 = 13.15$, $\Omega_2^2 = 5.64$)

3.10 Conservation of the Power Integral

In this section we shall investigate the assumption made by Hirota (24) which say that,

$$\int_{-\infty}^{\infty} P_n(\tau)\, d\tau = \text{constant, independent of position } n \qquad (3.11)$$

where,

$$P_n(\tau) = I_{n-1}(\tau)\; V_n(\tau) \qquad (3.12)$$

Two computer programs were written (see Appendix I and J) to compute the above integral for the case of two solitons moving in opposite directins and for the case of two solitons moving in the same direction

The results for the first case are tabulated in Table (3.12) for n=-7 to n=9. The results are in agreement with Hirota's assumption with a reasonable accuracy. The error between the absolute value of the integral of P_{-7} and the absolut value of the integral of P_9 is 0.00002.

The results for the second case are tabulated in Table (3.13). The results are again in agreement with Hirota's assumption with a maximum relative error of 0.02%.

n	$\int P_n(\tau)\, d\tau$
-7	-0.00250
-6	-0.00264
-5	-0.00264
-4	-0.00260
-3	-0.00260
-2	-0.00177
-1	-0.00083
0	-0.00037
1	0.00031
2	0.00070
3	0.00163
4	0.00244
5	0.00245
6	0.00247
7	0.00247
8	0.00234
9	0.00252

Table (3.12) Integral of $P_n(\tau)$ with respect to τ of two solitons moving in opposite directions for n=-7 to n=9.

n	$\int P_n(\tau)\, d\tau$
-7	16.15854
-6	16.15855
-5	16.15857
-4	16.15858
-3	16.15851
-2	16.15866
-1	16.15892
0	16.15887
1	16.15930
2	16.15987
3	16.15947
4	16.16113
5	16.16154
6	16.16158
7	16.16154
8	16.16151
9	16.16153

Table (3.13) Integral of $P_n(\tau)$ with respect to of two solitons moving in the same direction for n=-7 to n=9.

CHAPTER 4

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDIES

4.1 Conclusions

From the previous chapters it is seen that solitons occur in many different areas of physics and engineering namely,

(a) shallow water waves,

(b) magnetohydrodynamic waves in plasma,

(c) pressure waves in liquid-gas bubble mixture,

(d) ion acoustic waves in plasma,

(e) the anharmonic lattice,

(f) rotating fluid down a tube,

(g) nonlinear lumped networks.

The most interesting observation is that most of the above systems exhibit the same general solution. It is also worthwhile to notice that experimental observations verify the theoretical results of soliton characteristics.

For lattice solitons in nonlinear lumped networks we were able to verify the following properties:

(a) A single voltage soliton travels in the lumped nonlinear losseless network of Fig.(2.1) without changing shape.

(b) The amplitude of a single voltage soliton depends on the velocity of propagation.

(c) Two voltage solitons moving in opposite directions collide and emerge after collision without changing shapes.

(d) Two voltage solitons moving in the same direction with different velocities collide and emerge after collision without changing shapes.

(e) The integral of the voltage with respect to the normalized time is constant and indepependent of the position n for two voltage solitons moving in opposite directions and in the same direction.

(f) A single current soliton travels without changing shape.

(g) Two current solitons moving in opposite directions collide and emerge after collision without changing shapes.

(h) Two current solitons moving in the same direction with different velocities collide and emerge after collision without changing shapes.

(i) The integral of the power with respect to the normalized time for two solitons moving in opposite directions and in the same direction is constant and independent of the position n.

4.2 Recommendations for Further Investigations

To open the field for our followers regarding the investigation of solitons in nonlinear lumped networks we suggest the following:

(a) Applications of solitons in science and engineering should be sought.

(b) More experimental wrk should be applied to the nonlinear lumped network of Fig.(2.1).

(c) The above nonlinear lumped network should be simulated on a large analog computer to verify the solutions.

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Appendix A

Integral Solution

From Eq. (1.17)

$$\int_{\phi_o}^{\phi} \frac{d\phi}{\sqrt{P(\phi)}} = x - ut \tag{A.1}$$

But from Eq. (1.19)

$$P(\phi) \equiv u\phi^2 - \frac{\alpha}{3}\phi^3 \tag{A.2}$$

Substituting Eq. (A.2) into (A.1) gives

$$\int_{\phi_o}^{\phi} \frac{d\phi}{\phi\sqrt{u - \frac{\alpha}{3}\phi}} = x - ut \tag{A.3}$$

$$x - ut = \frac{2}{\sqrt{u}} \ln \frac{\sqrt{-\alpha/3\phi + u} - \sqrt{u}}{\sqrt{\phi}} \Bigg|_{\phi_o}^{\phi}$$

$$x\text{-}ut = \frac{2}{\sqrt{u}} \left[\ln \frac{\sqrt{-\alpha/3 + u} - \sqrt{u}}{\sqrt{\phi}} - \ln \frac{\sqrt{-\alpha/3\phi_o + u} - \sqrt{u}}{\sqrt{\phi_o}} \right]$$

$$\frac{\sqrt{u}}{2}(x\text{-}ut) = \ln \frac{-\sqrt{u}\left[- \sqrt{1 - \alpha\phi/3u} \right]}{\sqrt{u}\ \sqrt{3/\alpha}\ \sqrt{\alpha\phi/3u}} - \ln \frac{-\sqrt{u}\left[- \sqrt{1 - \alpha\phi_o/3u} \right]}{\sqrt{u}\sqrt{3/\alpha}\ \sqrt{\alpha\phi_o/3u}}$$

$$\frac{\sqrt{u}}{2}\ (x-ut) = \ln \frac{-\sqrt{\alpha/3}\ \left[1-\sqrt{\alpha\phi/3u}\right]}{\sqrt{\alpha\phi/3u}} \quad \ln \frac{-\sqrt{\alpha/3}\ \left[1-\sqrt{\alpha\phi_o/3u}\right]}{\sqrt{\alpha\phi_o/3u}}$$

$$\frac{\sqrt{u}}{2}\ (x-ut) = \ln - \sqrt{\alpha/3} + \ln \frac{1-\sqrt{\alpha\phi/3u}}{\sqrt{\alpha\phi/3u}} - \ln -\sqrt{\alpha/3}$$

$$- \ln \frac{1-\sqrt{\alpha\phi_o}}{\sqrt{\alpha\phi_o/3u}}$$

$$\frac{\sqrt{u}}{2}\ (x-ut) = \ln \frac{1-\sqrt{\alpha\phi/3u}}{\sqrt{\alpha\phi/3u}} - \ln \frac{1-\sqrt{\alpha\phi_o/3u}}{\sqrt{\alpha\phi/3u}} \tag{A.4}$$

But $\operatorname{sech}^{-1} x = \ln \frac{1-x}{x}$ (A.5)

$$\frac{\sqrt{u}}{2}\ (x-ut) = \operatorname{sech}^{-1} \sqrt{\alpha\phi/3u} - \operatorname{sech}^{-1} \sqrt{\alpha\phi_o/3u}$$

$$\operatorname{sech}^{-1} \sqrt{\alpha\phi/3u} = \frac{\sqrt{u}}{2}\ (x-ut) + \operatorname{sech}^{-1} \sqrt{3\phi_o/3u}$$

$$\sqrt{\alpha\phi/3u} = \operatorname{sech}\left[\frac{\sqrt{u}}{2}\ (x-ut) + \operatorname{sech}^{-1} \sqrt{3\phi_o/3u}\right]$$

$$\sqrt{\alpha\phi/3u} = \operatorname{sech}^2\left[\frac{\sqrt{u}}{2}\ (x-ut) + \operatorname{sech}^{-1} \sqrt{\alpha\phi_o/3u}\right]$$

$$\phi = 3u/\alpha\ \operatorname{sech}^2\left[\frac{\sqrt{u}}{2}\ (x-ut) + \operatorname{sech}^{-1} \sqrt{\alpha\phi_o/3u}\right]$$

$$\phi = 3u/\alpha \ \operatorname{sech}^2 \left[\frac{\sqrt{u}}{2} (x-ut) + \delta \right] \tag{A.5}$$

where

$$\delta = \operatorname{sech}^{-1} \sqrt{\alpha\phi_o/3u} \tag{A.6}$$

Appendix B

A computer program to calculate the magnitude of $V_n(\tau)/V_o$ of a single soliton for different nodes.

```
      N=-10
      P=1.999849137
      GAMSQ=13.15
      GAM=SQRT(GAMSQ)
      DO 500 I =1,21
      WRITE(6,2000)N,GAMSQ,P
      T=-10.0
      DO 600 J =1,201
      FUN=T*GAM-P*N
      VNT=GAMSQ/COSH(FUN)**2.
      WRITE(6,3000)T,VNT
      T=T+0.1
600   CONTINUE
500   N=N+1
2000  FORMAT('1',T20,I10,/,T20,F10.2,/,T20,F20.9,///)
3000  FORMAT(F10.3,E30.5)
      STOP
      END
```

Appendix C

A computer program to calculate the magnitude of $V_n(\tau)/V_o$ for three velocities and three nodes.

```
      DO 700 I =1,3
      N=-5
      READ(5,1000)GAMSQ,P
      GAM=SQRT(GAMSQ)
      DO 600 J =1,3
      WRITE(6,2000)N,GAMSQ,P
      T=-5.0
      DO 500 K =1,101
      FUN=T*GAM-P*N
      VNT=GAMSQ/COSH(FUN)**2.
      WRITE(6,3000)T,VNT
      T=T+0.1
500   CONTINUE
600   N=N+5
700   CONTINUE
1000  FORMAT(F6.2,F12.9)
2000  FORMAT('1',T20,I10,/,T20,F10.2,/,T20,F20.9,///)
3000  FORMAT(F10.1,E15.5)
      STOP
      END
```

Appendix D

A computer program to compute the magnitude of $V_n(\tau)/V_o$ for two solitons moving in opposite directions.

```
C.....THIS PROGRAM COMPUTE VN(TOW)/VO FOR TWO
C.....SOLITONS MOVING IN OPPOSITE DIRECTIONS
      N=-10
      GAM1SQ=13.15
      GAM2SQ=GAM1SQ
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=GAM1
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=-P1
      AN1=COSH((P1-P2)/2.)**2.
      AN2=COSH((P1+P2)/2.)**2.
      ANGL=0.25*ALOG(AN1/AN2)
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      DO 500 I =1,21
      WRITE(6,1000)N
      T=-10.0
      DO 600 J =1,200
      ATT1=GAM1*T-P1*N
      ATT2=GAM2*T-P2*N
      ANUM=(GAM1/COSH(ATT1))**2.+(GAM2/COSH(ATT2))**2.+AO/((COSH(ATT1)*C
     1OSH(ATT2))**2.)
      ADEN=(AA2+AA1*TANH(ATT1)*TANH(ATT2))**2.
      VNT=ANUM/ADEN
      IF(VNT.LT.0.04) GO TO 700
      WRITE(6,2000)T,VNT
700   T=T+0.1
600   CONTINUE
500   N=N+1
1000  FORMAT('1',' N= ',I10,///)
2000  FORMAT(F5.1,F10.4)
      STOP
      END
```

Appendix E

A computer program to compute the magnitude of $V_n(\tau)/V_o$ for two solitons moving in the same direction.

```
C.....THIS PROGRAM COMPUTE VN(TOW)/VO FOR TWO
C.....SOLITONS MOVING IN THE SAME DIRECTION
      N=-10
      GAM1SQ=13.15
      GAM2SQ=5.64
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=SQRT(GAM2SQ)
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=1.599728561
      AN1=SINH((P1-P2)/2.)**2.
      AN2=SINH((P1+P2)/2.)**2.
      ANGL=0.25*ALOG(AN1/AN2)
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      DO 500 I =1,18
      WRITE(6,1000)N
      T=-10.0
      DO 600 J =1,188
      ATT1=GAM1*T-P1*N
      ATT2=GAM2*T-P2*N
      ANUM=(GAM1/COSH(ATT1))**2.+(GAM2/COSH(ATT2))**2.+AO/((COSH(ATT1)*C
     1OSH(ATT2))**2.)
      ADEN=(AA2+AA1*TANH(ATT1)*TANH(ATT2))**2.
      VNT=ANUM/ADEN
      IF(VNT.LT.0.04) GO TO 700
      WRITE(6,2000)T,VNT
700   T=T+0.1
600   CONTINUE
500   N=N+1
1000  FORMAT('1','  N=  ',I10,///)
2000  FORMAT(F5.1,F10.4)
      STOP
      END
```

Appendix F

A computer program to compute the integral of the voltage with respect to tow of two solitons moving in the same direction

```
      DIMENSION TT(2002),AVNT(2002)
      WRITE(6,1000)
      N=-10
      GAM1SQ=13.15
      GAM2SQ=5.64
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=SQRT(GAM2SQ)
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=1.599728561
      AN1=SINH((P1-P2)/2.)**2.
      AN2=SINH((P1+P2)/2.)**2.
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      DO 500 I =1,18
      T=-10.0
      DO 600 J =1,1871
      ATT1=GAM1*T-P1*N
      ATT2=GAM2*T-P2*N
      ANUM=(GAM1/COSH(ATT1))**2.+(GAM2/COSH(ATT2))**2.+AO/((COSH(ATT1)*C
     1OSH(ATT2))**2.)
      ADEN=(AA2+AA1*TANH(ATT1)*TANH(ATT2))**2.
      VNT=ANUM/ADEN
      AVNT(J)=VNT
      T=T+0.01
      TT(J+1)=T
600   CONTINUE
      TT(1)=-10.0
      SUM2=0.0
      DO 700 K =2,1871
      SUM1=SUM2
      SUM2=SUM1+0.5*(TT(K)-TT(K-1))*(AVNT(K)+AVNT(K-1))
700   CONTINUE
      WRITE(6,4000)N,SUM2
500   N=N+1
      WRITE(6,2000)
1000  FORMAT('1','  INTEGRAL OF VN(TOW)/VO WITH RESPECT TO TOW',/,'  OF
     1TWO SOLITONS MOVING IN THE SAME DIRECTION',/,'   FOR N=-10 TO N=7'
     2,//)
2000  FORMAT('1')
4000  FORMAT('   N=  ',I3,5X,' VOLTAGE INTEGRAL= ',E13.5,/)
      STOP
      END
```

Appendix G

A computer program to compute the integral of the voltage with respect to tow of two solitons moving in opposite directions.

```
      DIMENSION TT(2002),AVNT(2002)
      WRITE(6,1000)
      N=-10
      GAM1SQ=13.15
      GAM2SQ=GAM1SQ
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=SQRT(GAM2SQ)
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=-P1
      AN1=COSH((P1-P2)/2.)**2.
      AN2=COSH((P1+P2)/2.)**2.
      ANGL=0.25*ALOG(AN1/AN2)
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      DO 500 I =1,21
      T=-10.0
      DO 600 J =1,2001
      ATT1=GAM1*T-P1*N
      ATT2=GAM2*T-P2*N
      ANUM=(GAM1/COSH(ATT1))**2.+(GAM2/COSH(ATT2))**2.+AO/((COSH(ATT1)*C
     1OSH(ATT2))**2.)
      ADEN=(AA2+AA1*TANH(ATT1)*TANH(ATT2))**2.
      VNT=ANUM/ADEN
      AVNT(J)=VNT
      T=T+0.01
      TT(J+1)=T
600   CONTINUE
      TT(1)=-10.0
      SUM2=0.0
      DO 700 K =2,2001
      SUM1=SUM2
      SUM2=SUM1+0.5(TT(K)-TT(K-1))*(AVNT(K)+AVNT(K-1))
700   CONTINUE
      WRITE(6,4000)N,SUM2
500   N=N+1
      WRITE(6,2000)
1000  FORMAT('1','  INTEGRAL OF VN(TOW)/VO WITH RESPECT TO TOW',/,'  OF
     1TWO SOLITONS MOVING IN OPPOSITE DIRECTIONS',/,'  FOR N=-10 TO N=10
     2',//)
2000  FORMAT('1')
4000  FORMAT('   N=  ',I3,5X,' VOLTAGE INTEGRAL= ',E13.5,/)
      STOP
      END
```

Appendix H

A computer program to calculate the magnitude of $I_n(\tau)$ of a single soliton for different nodes.

```
      DIMENSION DIFF(2001),CURR(2001)
      N=-10
      P=1.999849137
      GAMSQ=13.15
      GAM=SQRT(GAMSQ)
      T=-10.0
      DO 600 I =1,21
      WRITE(6,2000)N
      MM=N+1
      DO 500 J =1,2001
      FUN1=T*GAM-P*N
      FUN2=T*GAM-P*MM
      VNT1=GAMSQ/(COSH(FUN1))**2.
      VNT2=GAMSQ/(COSH(FUN2))**2.
      DIFF(J)=VNT1-VNT2
      T=T+0.01
500   CONTINUE
      CURR(1)=0.0
      TT=-10.0
      PRINT3000,TT,CURR(1)
      DO 550 K =2,2001
      CURR(K)=CURR(K-1)+0.005*(DIFF(K)+DIFF(K-1))
      IF(MOD(K,10).EQ.0)PRINT3000,TT,CURR(K)
      TT=TT+0.01
550   CONTINUE
      N=N+1
600   T=-10.0
3000  FORMAT(5X,F5.1,5X,F15.5)
2000  FORMAT('1','N=  ',I3)
      STOP
      END
```

Appendix I

A computer program to calculate the magnitude of $I_n(\tau)$ of two current solitons moving in opposite directions and to calculate the integral of the power with respect to tow for different nodes.

```
      DIMENSION DIFF(1801),CURR(1801),VOLT(1801),POWER(1801)
      N=-8
      KKKK=17
      NNNN=1801
      GAM1SQ=13.15
      GAM2SQ=GAM1SQ
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=GAM1
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=-P1
      AN1=(COSH((P1-P2)/2.))**2.
      AN2=(COSH((P1+P2)/2.))**2.
      ANGL=0.25*ALOG(AN1/AN2)
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      WRITE(6,6000)
      DO 700 I =1,KKKK
      WRITE(6,4000)N
      M=N+1
      T=-9.0
      DO 600 J =1,NNNN
      ATT11=GAM1*T-P1*N
      ATT12=GAM2*T-P2*N
      ATT21=GAM1*T-P1*M
      ATT22=GAM2*T-P2*M
      CONTINUE
      ANUM=(GAM1/COSH(ATT11))**2.+(GAM2/COSH(ATT12))**2.+AO/((COSH(ATT11
     1)*COSH(ATT12))**2.)
      ADEN1=(AA2+AA1*TANH(ATT11)*TANH(ATT12))**2.
      VNT1=ANUM/ADEN1
      ANUM2=(GAM1/COSH(ATT21))**2.+(GAM2/COSH(ATT22))**2.+AO/((COSH(ATT2
     11)*COSH(ATT22))**2.)
      ADEN2=(AA2+AA1*TANH(ATT21)*TANH(ATT22))**2.
      VNT2=ANUM2/ADEN2
      VOLT(J)=VNT2
      DIFF(J)=VNT1-VNT2
600   T=T+0.01
```

```
      CURR(1)=0.0
      TT=-9.0
      DO 650 K =2,NNNN
      CURR(K)=CURR(K-1)+0.005*(DIFF(K-1)+DIFF(K))
      IF(MOD(K,10).EQ.0)PRINT3000,TT,CURR(K)
      TT=TT+0.01
650   CONTINUE
      DO 660 L =1,NNNN
660   POWER(L)=CURR(L)*VOLT(L)
      SUM1=0.0
      DO 670 LL =1,NNNN
670   SUM1=SUM1+0.005*(POWER(LL)+POWER(L+1)
      WRITE(6,5000)M,SUM1
      TT=-9.0
      N=N+1
700   T=-9.0
3000  FORMAT(F8.1,F10.5)
4000  FORMAT('1',5X,'TOW',10X,'I(',I3,')',///)
6000  FORMAT('1')
5000  FORMAT(I10,F15.5)
      STOP
      END
```

Appendix J

A computer program to calculate the magnitude of $I_n(\tau)$ of two current solitons moving in the same direction and to calculate the integral of the power with respect to tow for different nodes

```
      DIMENSION DIFF(1801),CURR(1801),VOLT(1801),POWER(1801)
      N=-8
      KKKK=17
      NNNN=1801
      GAM1SQ=13.15
      GAM2SQ=5.64
      SUM=GAM1SQ+GAM2SQ
      GAM1=SQRT(GAM1SQ)
      GAM2=SQRT(GAM2SQ)
      PROD=GAM1*GAM2*2.
      P1=1.999849137
      P2=1.599728561
      AN1=(SINH((P1-P2)/2.))**2.
      AN2=(SINH((P1+P2)/2.))**2.
      ANGL=0.25*ALOG(AN1/AN2)
      AA1=SINH(ANGL)
      AA2=COSH(ANGL)
      AO=AA1*(SUM*AA1+PROD*AA2)
      WRITE(6,6000)
      DO 700 I =1,KKKK
      M=N+1
      T=-9.0
      DO 600 J =1,NNNN
      ATT11=GAM1*T-P1*N
      ATT12=GAM2*T-P2*N
      ATT21=GAM1*T-P1*M
      ATT22=GAM2*T-P2*M
      ANUM=(GAM1/COSH(ATT11))**2.+(GAM2/COSH(ATT12))**2.+AO/((COSH(ATT11
     1)*COSH(ATT12))**2.)
      ADEN1=(AA2+AA1*TANH(ATT11)*TANH(ATT12))**2.
      VNT1=ANUM/ADEN1
      ANUM2=(GAM1/COSH(ATT21))**2.+(GAM2/COSH(ATT22))**2.+AO/((COSH(ATT2
     11)*COSH(ATT22))**2.)
      ADEN2=(AA2+AA1*TANH(ATT21)*TANH(ATT22))**2.
      VNT2=ANUM2/ADEN2
      VOLT(J)=VNT1-VNT2
600   T=T+0.01
      CURR(1)=0.0
      TT=-9.0
650   CONTINUE
      DO 660 L =1,NNNN
660   POWER(L)=CURR(L)*VOLT(L)
      SUM1=0.0
```

```
      DO 670 LL =1,NNNN
670   SUM1=SUM1+0.005*(POWER(LL)+POWER(L+1))
      WRITE(6,5000)M,SUM1
      N=N+1
700   T=-9.0
3000  FORMAT(F8.1,F10.5)
4000  FORMAT('1',5X,'TOW',10X,'I(',I3,')',///)
5000  FORMAT(I10,F15.5)
6000  FORMAT('1')
      STOP
      END
```